ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-863 Proposed by Kenneth B. Davenport, Dallas, PA

Show that

$$\sum_{n\geq 1}\frac{\zeta(2n+1)-1}{2n+1}=1-\gamma-\frac{\ln 2}{2}\quad\text{and}\quad\sum_{n\geq 1}\frac{\zeta(2n)-1}{n(n+1)}=\ln(2\pi)-\frac{3}{2},$$

where $\zeta(n)$ is the Riemann zeta function.

H-864 Proposed by Hideyuki Ohtsuka, Saitama, Japan

The Pell numbers $\{P_n\}_{n\geq 0}$ satisfy $P_0=0$, $P_1=1$, and $P_n=2P_{n-1}+P_{n-2}$ for $n\geq 2$. Prove that

$$\sum_{n=1}^{\infty}\tan^{-1}\frac{1}{\sqrt{2}P_n}\tan^{-1}\frac{1}{\sqrt{2}P_{n+1}}=\frac{\pi}{4}\tan^{-1}\frac{1}{2\sqrt{2}}.$$

H-865 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

Let $\{x_n\}_{n\geq 0}$ be the sequence given by $x_0=0$, $x_1=1$, and

$$x_{n+2}=(2n+5)x_{n+1}-(n^2+4n+3)x_n\quad\text{for}\quad n\geq 0.$$

Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}}-\sqrt[n]{F_nL_nx_n}\right).$$

H-866 Proposed by Ángel Plaza, Gran Canaria, Spain

Let a_n denote the nth number in the sequence given by $a_{n+1}=a_n+a_{n-1}$ for $n\geq 1$ with initial values $a_0=a-1$ and $a_1=1$ with some $a\geq 1$. Prove that

$$\sum_{k=1}^{n}\frac{2(a_{k+1}-a_k)}{a_{k+1}+a_k}<\ln a_{n+1}<\sum_{k=1}^{n}\frac{a_{k+1}^2-a_k^2}{2a_{k+1}a_k}.$$

H-867 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let a, b, c, d be even positive integers with $a+b=c+d$. Prove that

$$\sum_{k=1}^{a}\frac{L_b}{F_kL_{k+b}}+\sum_{k=1}^{b}\frac{L_a}{L_kF_{k+a}}=\sum_{k=1}^{c}\frac{L_d}{F_kL_{k+d}}+\sum_{k=1}^{d}\frac{L_c}{L_kF_{k+c}}.$$

SOLUTIONS

A sum of arctangents

H-829 Proposed by Ángel Plaza and Francisco Perdomo, Gran Canaria, Spain (Vol. 56, No. 4, November 2018)

For any positive integer k, let $\{F_{k,n}\}_{n\geq 0}$ be the sequence defined by $F_{k,0}=0$, $F_{k,1}=1$, and $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$ for $n\geq 1$. Find the limit

$$\lim_{k\to\infty}\frac{k+\sqrt{k^2+4}}{2}\sum_{n=1}^{\infty}\arctan\left(\frac{kF_{k,n+1}^2}{1+F_{k,n}F_{k,n+1}^2F_{k,n+2}}\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland

We note that

$$\begin{aligned}\arctan\frac{1}{F_{k,n}F_{k,n+1}}-\arctan\frac{1}{F_{k,n+1}F_{k,n+2}} &= \arctan\frac{\frac{1}{F_{k,n}F_{k,n+1}}-\frac{1}{F_{k,n+1}F_{k,n+2}}}{1+\frac{1}{F_{k,n}F_{k,n+1}^2F_{k,n+2}}}\\ &= \arctan\frac{F_{k,n+1}(F_{k,n+2}-F_{k,n})}{1+F_{k,n}F_{k,n+1}^2F_{k,n+2}}\\ &= \arctan\frac{kF_{k,n+1}^2}{1+F_{k,n}F_{k,n+1}^2F_{k,n+2}}.\end{aligned}$$

So,

$$\sum_{n=1}^{\infty}\arctan\frac{kF_{k,n+1}^2}{1+F_{k,n}F_{k,n+1}^2F_{k,n+2}}=\arctan\frac{1}{F_{k,1}F_{k,2}}=\arctan\frac{1}{k},$$

and

$$\lim_{k\to\infty}\frac{k+\sqrt{k^2+4}}{2}\sum_{n=1}^{\infty}\arctan\frac{kF_{k,n+1}^2}{1+F_{k,n}F_{k,n+1}^2F_{k,n+2}}=\lim_{k\to\infty}\frac{k+\sqrt{k^2+4}}{2}\arctan\frac{1}{k}=1.$$

Also solved by Brian Bradie, Dmitry Fleischman, Robert Frontczak, and the proposers.

A sum divisible by four consecutive Fibonacci numbers

H-830 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 56, No. 4, November 2018)

For an integer $n \geq 1$, prove that

$$12\sum_{k=1}^{n}(F_kF_{k+1}F_{k+2})^2 \equiv 0 \pmod{F_nF_{n+1}F_{n+2}F_{n+3}}.$$

Solution by the proposer

Using $F_{a+b}F_{a+c} = F_aF_{a+b+c} + (-1)^aF_bF_c$ (see [3] (20a)), we have

$$F_kF_{k+2} = F_{k-1}F_{k+3} + (-1)^{k-1}F_1F_3 = F_{k-1}F_{k+2} - 2(-1)^k. \tag{1}$$

We have

$$\begin{aligned}\sum_{k=1}^{n}(F_kF_{k+1}F_{k+2})^2 &= \sum_{k=1}^{n}(F_kF_{k+1}^2F_{k+2}) \times (F_kF_{k+2}) \\ &= \sum_{k=1}^{n}F_kF_{k+1}^2F_{k+2}(F_{k-1}F_{k+3} - 2(-1)^k) \qquad \text{by (1)} \\ &= \sum_{k=1}^{n}F_{k-1}F_kF_{k+1}^2F_{k+2}F_{k+3} - 2\sum_{k=1}^{n}(-1)^kF_kF_{k+1}^2F_{k+2}.\end{aligned} \tag{2}$$

From identity (2.1) in [1], we have

$$\sum_{k=1}^{n}F_{k-1}F_kF_{k+1}^2F_{k+2}F_{k+3} = \frac{1}{4}F_{n-1}F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}. \tag{3}$$

From identity (2.17) in [2], we have

$$\sum_{k=1}^{n}(-1)^kF_kF_{k+1}^2F_{k+2} = \frac{1}{3}(-1)^nF_nF_{n+1}F_{n+2}F_{n+3}. \tag{4}$$

By (2), (3), and (4), we have

$$\begin{aligned}12\sum_{k=1}^{n}(F_kF_{k+1}F_{k+2})^2 &= F_nF_{n+1}F_{n+2}F_{n+3}(3F_{n-1}F_{n+4} - 8(-1)^n) \\ &\equiv 0 \pmod{F_nF_{n+1}F_{n+2}F_{n+3}}.\end{aligned}$$

[1] R. S. Melham, *Sums of certain products of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **37.3** (1999), 248–251.
[2] R. S. Melham, *Sums of certain products of Fibonacci and Lucas numbers, II*, The Fibonacci Quarterly **38.1** (2000), 3–7.
[3] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Also solved by Kenneth B. Davenport and Raphael Schumacher.

Proth primality test using Fibonacci numbers

H-831 Proposed by Predrag Terzić, Podgorica, Montenegro (Vol. 56, No. 4, November 2018)

Let $P_j(x) = 2^{-j}((x-\sqrt{x^2-4})^j + (x+\sqrt{x^2-4})^j)$, where j and x are nonnegative integers. Let $N = k2^m + 1$ with k odd, $k < 2^m$, and $m > 2$. Let $S_0 = P_k(F_n)$ and $S_i = S_{i-1}^2 - 2$ for $i \geq 1$. Prove the following statement: If there exists F_n for which $S_{m-2} \equiv 0 \pmod{N}$, then N is prime.

No solution to this problem was received. The proposer pointed out [1], where some particular cases are treated (the cases $n = 4, 5, 6$ and k and m in various residue classes).

[1] P. Terzić, *Primality tests for specific classes of* $N = k2^m \pm 1$, arXiv: 1506.03444v1(2015).

Closed form expressions for sums with Fibonacci and Lucas numbers

H-832 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 56, No. 4, November 2018)

For positive integers n and r, find a closed form expression for

(i) $\sum_{k=1}^{n} F_{rk}^3 L_{rk}$;
(ii) $\sum_{k=1}^{n} F_{2F_k}^3 F_{2L_k}$.

Solution by the proposer

We use Catalan's identity

$$F_n^2 - (-1)^{n-m} F_m^2 = F_{n+m} F_{n-m}. \tag{5}$$

(i) We have

$$\begin{aligned} F_{2r} \sum_{k=1}^{n} F_{rk}^3 L_{rk} &= \sum_{k=1}^{n} F_{rk}^2 (F_{2kr} F_{2r}) \\ &= \sum_{k=1}^{n} F_{rk}^2 (F_{r(k+1)}^2 - F_{r(k-1)}^2) \quad \text{by (5)} \\ &= \sum_{k=1}^{n} \left(F_{rk}^2 F_{r(k+1)}^2 - F_{r(k-1)}^2 F_{rk}^2 \right) \\ &= F_{rn}^2 F_{r(n+1)}^2. \end{aligned}$$

Thus, we obtain

$$\sum_{k=1}^{n} F_{rk}^3 L_{rk} = \frac{F_{rn}^2 F_{r(n+1)}^2}{F_{2r}}.$$

(ii) We have

$$\begin{aligned}\sum_{k=1}^{n} F_{2F_k}^3 F_{2L_k} &= \sum_{k=1}^{n} F_{2F_k}^2 (F_{2F_k} F_{2L_k}) \\ &= \sum_{k=1}^{n} F_{2F_k}^2 (F_{F_k+L_k}^2 - F_{F_k-L_k}^2) \\ &= \sum_{k=1}^{n} F_{2F_k}^2 (F_{2F_{k+1}}^2 - F_{-2F_{k-1}}^2) \quad (\text{since } L_k = F_{k-1} + F_{k+1}) \\ &= \sum_{k=1}^{n} (F_{2F_k}^2 F_{2F_{k+1}}^2 - F_{2F_{k-1}}^2 F_{2F_k}^2) = F_{2F_n}^2 F_{2F_{n+1}}^2.\end{aligned}$$

Also solved by Brian Bradie, Dmitry Fleischman, Robert Frontczak, and Raphael Schumacher.

Closed form for a sum of Tribonacci Lucas numbers

H-833 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 57, No. 1, February 2019)

The Tribonacci-Lucas numbers $\{K_n\}_{n\geq 0}$ satisfy $K_0 = 3$, $K_1 = 1$, $K_2 = 3$, and $K_n = K_{n-1} + K_{n-2} + K_{n-3}$ for $n \geq 3$. Prove that for any $n \geq 1$

$$\sum_{j=1}^{n} K_{2j} K_{2j+1} = \frac{1}{4}((K_{2n} + K_{2n+1})^2 - 16).$$

Solution by Brian Bradie, Newport News, VA

Observe

$$\begin{aligned}(K_{2j} + K_{2j+1})^2 - (K_{2j-2} + K_{2j-1})^2 &= (K_{2j} + K_{2j+1} + K_{2j-2} + K_{2j-1}) \\ &\times (K_{2j} + K_{2j+1} - K_{2j-2} - K_{2j-1}) \\ &= (2K_{2j+1})(2K_{2j}) = 4K_{2j}K_{2j+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{n} K_{2j} K_{2j+1} &= \frac{1}{4} \sum_{j=1}^{n} ((K_{2j} + K_{2j+1})^2 - (K_{2j-2} + K_{2j-1})^2) \\ &= \frac{1}{4}((K_{2n} + K_{2n+1})^2 - (K_0 + K_1)^2) \\ &= \frac{1}{4}((K_{2n} + K_{2n+1})^2 - 16).\end{aligned}$$

Also solved by Kenneth B. Davenport, Wei-Kai Lai and John Risher (jointly), Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher, David Terr, and the proposer.

Late acknowledgement: Albert Stadler has solved Advanced Problem **H-825**.