ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

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PROBLEMS PROPOSED IN THIS ISSUE

H-883 Proposed by Kenneth B. Davenport, Dallas, PA

Prove that for all $n \geq 1$:

(a) $3\sum_{k=1}^{n} F_{2k} + 4\sum_{k=1}^{n} F_{2k}^3 = F_{2n+1}^3 - 1;$

(b) $5\sum_{k=1}^{n} F_{2k} + 15\sum_{k=1}^{n} F_{2k}^3 + 11\sum_{k=1}^{n} F_{2k}^5 = F_{2n+1}^5 - 1;$

(c) $7\sum_{k=1}^{n} F_{2k} + 35\sum_{k=1}^{n} F_{2k}^3 + 56\sum_{k=1}^{n} F_{2k}^5 + 29\sum_{k=1}^{n} F_{2k}^7 = F_{2n+1}^7 - 1.$

H-884 Proposed by Robert Frontczak, Stuttgart, Germany

Prove that

(i) $\sum_{n=2}^{\infty} \coth^{-1}(\alpha^n - \alpha^{-n}) = \frac{1}{2}\ln((\alpha+1)(\alpha+2)), \quad \sum_{n=1}^{\infty} \coth^{-1}(\alpha^{2n} - \alpha^{-2n}) = \frac{1}{2}\ln(\alpha^3),$
and $\sum_{n=1}^{\infty} \coth^{-1}(\alpha^{2n+1} - \alpha^{-2n-1}) = \frac{1}{2}\ln\left(\frac{\alpha+2}{\alpha}\right).$

(ii) Deduce from (a) the following series evaluations
$\sum_{n=1}^{\infty} \coth^{-1}\left(\frac{L_{4n+2}-1}{2L_{2n+1}}\right) = \ln\left(\frac{\alpha+2}{\alpha}\right), \quad \sum_{n=1}^{\infty} \coth^{-1}\left(\frac{L_{4n}-1}{2\sqrt{5}F_{2n}}\right) = 3\ln\alpha,$
and $\sum_{n=2}^{\infty} \coth^{-1}\left(\beta^n - \beta^{-n}\right) = \frac{1}{2}\ln((\alpha+1)(\alpha+2)) - 3\ln\alpha.$

H-885 Proposed by Robert Frontczak, Stuttgart, Germany

Show that

$$\sum_{i=1}^{\infty} H_{2i-r}^{(2)} \frac{1}{\alpha^{2i}} = \left(\frac{\alpha+5-r}{10}\right)\frac{\pi^2}{6} - \left(\frac{\alpha+3-r}{4}\right)\ln^2(\alpha) \quad \text{hold for} \quad r = 0, 1,$$

where $H_n^{(2)} = \sum_{m=1}^{n} 1/m^2$. Deduce from these two identities the known (but nontrivial) result

$$\sum_{i=1}^{\infty} \frac{1}{i^2\alpha^{2i}} = \frac{\pi^2}{15} - \ln^2(\alpha).$$

H-886 Proposed by D. M. Bătineţu–Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

If $a, b, c \in (0, \pi/2)$ and $n \geq 1$, prove that

$$\text{(i)} \quad \frac{\tan a}{F_n \sin 2b + F_{n+1} \sin 2c} + \frac{\tan b}{F_n \sin 2c + F_{n+1} \sin 2a} + \frac{\tan c}{F_n \sin 2a + F_{n+1} \sin 2b} > \frac{3}{2F_{n+2}};$$

$$\text{(ii)} \quad \frac{\tan a}{F_n^2 \sin 2b + F_{n+1}^2 \sin 2c} + \frac{\tan b}{F_n^2 \sin 2c + F_{n+1}^2 \sin 2a} + \frac{\tan c}{F_n^2 \sin 2a + F_{n+1}^2 \sin 2b} > \frac{3}{2F_{2n+1}}.$$

H-887 Proposed by D. M. Bătineţu–Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

If $m \geq 1$ is an integer, compute $\lim_{n\to\infty} n^{\cos^2 F_m}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{\sin^2 F_m} - \left(\sqrt[n]{n!}\right)^{\sin^2 F_m}\right)$.

H-888 Proposed by José Luis Díaz–Barrero, Barcelona, Spain

For any integer $n \geq 1$, prove that

$$\sqrt{6F_n^4 + 3L_n^4} + \sqrt{5F_n^4 + 4L_n^4} + \sqrt{7F_n^4 + 2L_n^4} \geq F_{n+3}^2.$$

SOLUTIONS

An identity with multinomial coefficients and Lucas numbers

H-849 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 57, No. 4, November 2019)

For nonnegative integers m and n, find a closed form formula for

$$\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} (-1)^j L_{i-j} \binom{m}{k} \binom{n}{i,j,k}.$$

Solution by Albert Stadler, Herrliberg, Switzerland

We note that

$$(-1)^j L_{i-j} = (-1)^j(\alpha^{i-j} + \beta^{i-j}) = \alpha^i\beta^j + \alpha^j\beta^i.$$

Therefore,

$$\begin{aligned}\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}}\binom{n}{i,j,k}(-1)^j L_{i-j} z^k &= \sum_{\substack{i+j+k=n\\ i,j,k\geq 0}}\binom{n}{i,j,k}(\alpha^i\beta^j+\alpha^j\beta^i)z^k\\ &= (\alpha+\beta+z)^n+(\beta+\alpha+z)^n = 2(1+z)^n\\ &= 2\sum_{k=0}^{n}\binom{n}{k}z^k,\end{aligned}$$

and by identifying coefficients, we conclude that for a fixed $k \in \{0, 1, \ldots, n\}$, we have

$$\sum_{\substack{i+j+k=n\\ i,j\geq 0}}\binom{n}{i,j,k}(-1)^j L_{i-j} = 2\binom{n}{k}.$$

We deduce that

$$\begin{aligned}\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}}\binom{n}{i,j,k}(-1)^j L_{i-j}\binom{m}{k} &= \sum_{k=0}^{n}\binom{m}{k}\sum_{\substack{i+j+k=n\\ i,j\geq 0}}\binom{n}{i,j,k}(-1)^j L_{i-j}\\ &= 2\sum_{k=0}^{n}\binom{m}{k}\binom{n}{n-k} = 2\binom{m+n}{n}.\end{aligned}$$

Also solved by Brian Bradie, Dmitry Fleischman, Raphael Schumacher, and the proposer.

A formula for the area of a triangle with Fibonacci coordinates

H-850 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 58, No. 1, February 2019)

For integers m, n, r, and s, let

$$\vec{AB} = (F_m, F_{m+r}, F_{m+s}) \quad \text{and} \quad \vec{AC} = (F_n, F_{n+r}, F_{n+s}).$$

Prove that the area of the triangle ABC is

$$\frac{1}{2}\sqrt{F_r^2+F_s^2+F_{r-s}^2}\,|F_{n-m}|.$$

Solution by Steve Edwards, Roswell, GA

Because such an area is given by one-half the magnitude of the cross product of the two vectors, the area equals

$$\frac{1}{2}\sqrt{(F_{m+r}F_{n+s}-F_{n+r}F_{m+s})^2+(F_mF_{n+s}-F_nF_{m+s})^2+(F_mF_{n+r}-F_nF_{m+r})^2}.$$

Each of the three squared differences under the radical can be transformed by using the identity $F_{a+b}F_{a+c} - F_aF_{a+b+c} = (-1)^a F_bF_c$, which can be found in [1], giving

$$\frac{1}{2}\sqrt{F_{s-r}^2F_{n-m}^2+F_s^2F_{n-m}^2+F_r^2F_{n-m}^2} = \frac{1}{2}\sqrt{F_r^2+F_s^2+F_{r-s}^2}\,|F_{n-m}|.$$

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, 2nd ed., John Wiley and Sons, 2018.

Also solved by Michel Bataille, Alan Duan, Brian Bradie, Dmitry Fleischman, G. C. Greubel, Wei-Kai Lai, Kapil Kumar Gurjar, Alejandro Pinilla-Barrera, Raphael Schumacher, Jason Smith, Albert Stadler, and the proposer.

A limit with nth roots of F_n

H-851 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania (Vol. 58, No. 1, February 2019)

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of positive real numbers such that $\lim_{n\to\infty} a_{n+1}/(n^r a_n) = a \in \mathbb{R}_+^*$ and $\lim_{n\to\infty} b_{n+1}/(n^s b_n) = b \in \mathbb{R}_+^*$, where $r, s \in \mathbb{R}_+$. Compute

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{F_{n+1}}}{n^{r+s}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{F_n}}{(n+1)^{r+s}}\right)\sqrt[n]{b_n}.$$

Solution by Brian Bradie, Newport News, VA

Note

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^r} &= \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{rn}}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{r(n+1)}}\frac{n^{rn}}{a_n}\\ &= \lim_{n\to\infty}\frac{a_{n+1}}{n^r a_n}\left(\frac{n}{n+1}\right)^{r(n+1)}=\frac{a}{e^r}.\end{aligned}$$

Similarly,

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^s}=\frac{b}{e^s}.$$

With

$$\lim_{n\to\infty}\sqrt[n]{F_n}=\lim_{n\to\infty}\sqrt[n+1]{F_{n+1}}=\alpha,$$

it follows that

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{F_{n+1}}}{n^{r+s}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{F_n}}{(n+1)^{r+s}}\right)\sqrt[n]{b_n}\\ &= \lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}}{n^r}\cdot\sqrt[n+1]{F_{n+1}}-\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^r}\cdot\sqrt[n]{F_n}\left(\frac{n}{n+1}\right)^s\right)\frac{\sqrt[n]{b_n}}{n^s}\\ &= \left(\frac{a}{e^r}\cdot\alpha-\frac{a}{e^r}\cdot\alpha\cdot 1\right)\frac{b}{e^s}=0.\end{aligned}$$

Consider the slightly modified question: compute

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{F_{n+1}}}{n^{r+s-1}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{F_n}}{(n+1)^{r+s-1}}\right)\sqrt[n]{b_n}.$$

Working as above,

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt[n]{a_n/F_n}}{n^r} &= \lim_{n\to\infty}\sqrt[n]{\frac{a_n/F_n}{n^{rn}}}=\lim_{n\to\infty}\frac{a_{n+1}/F_{n+1}}{(n+1)^{r(n+1)}}\frac{n^{rn}}{a_n/F_n}\\ &= \lim_{n\to\infty}\frac{a_{n+1}}{n^r a_n}\cdot\frac{F_n}{F_{n+1}}\left(\frac{n}{n+1}\right)^{r(n+1)}=\frac{a}{\alpha e^r}.\end{aligned}$$

Let

$$u_n = \frac{\sqrt[n+1]{a_{n+1}/F_{n+1}}}{\sqrt[n]{a_n/F_n}}.$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} u_n &= \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}/F_{n+1}}}{(n+1)^r} \cdot \frac{n^r}{\sqrt[n]{a_n/F_n}} \left(\frac{n+1}{n}\right)^r \\
&= \frac{a}{\alpha e^r} \cdot \frac{\alpha e^r}{a} \cdot 1 = 1, \\
\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} &= 1, \text{ and} \\
\lim_{n\to\infty} u_n^n &= \lim_{n\to\infty} \frac{a_{n+1}}{n^r a_n} \frac{F_n}{F_{n+1}} \frac{(n+1)^r}{\sqrt[n+1]{a_{n+1}/F_{n+1}}} \left(\frac{n}{n+1}\right)^r \\
&= a \cdot \frac{1}{\alpha} \cdot \frac{\alpha e^r}{a} \cdot 1 = e^r,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}/F_{n+1}} - \sqrt[n]{a_n/F_n}}{n^{r-1}} &= \lim_{n\to\infty} \frac{\sqrt[n]{a_n/F_n}}{n^{r-1}}(u_n - 1) \\
&= \lim_{n\to\infty} \frac{\sqrt[n]{a_n/F_n}}{n^r} \frac{u_n - 1}{\ln u_n} \ln u_n^n \\
&= \frac{a}{\alpha e^r} \cdot 1 \cdot r = \frac{ar}{\alpha e^r}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{\sqrt[n]{a_n} \cdot \sqrt[n+1]{F_{n+1}}}{n^{r+s-1}} - \frac{\sqrt[n+1]{a_{n+1}} \cdot \sqrt[n]{F_n}}{(n+1)^{r+s-1}} \right) \sqrt[n]{b_n} \\
&= \lim_{n\to\infty} \sqrt[n+1]{F_{n+1}} \sqrt[n]{F_n} \left(\frac{\sqrt[n]{a_n/F_n}}{n^{r-1}} - \frac{\sqrt[n+1]{a_{n+1}/F_{n+1}}}{n^{r-1}} \frac{1}{(1+1/n)^{r+s-1}} \right) \frac{\sqrt[n]{b_n}}{n^s} \\
&= \lim_{n\to\infty} \sqrt[n+1]{F_{n+1}} \sqrt[n]{F_n} \left(\frac{\sqrt[n]{a_n/F_n} - \sqrt[n+1]{a_{n+1}/F_{n+1}}}{n^{r-1}} \right. \\
&\qquad \left. +(r+s-1) \frac{\sqrt[n+1]{a_{n+1}/F_{n+1}}}{(n+1)^r} \cdot \frac{(n+1)^r}{n^r} + O\left(\frac{1}{n}\right) \right) \frac{\sqrt[n]{b_n}}{n^s} \\
&= \alpha^2 \left(-\frac{ar}{\alpha e^r} + (r+s-1) \frac{a}{\alpha e^r} \right) \frac{b}{e^s} = \frac{ab(s-1)\alpha}{e^{r+s}}.
\end{aligned}$$

Also solved by Michel Bataille, Dmitry Fleischman, Raphael Schumacher, and the proposers.

A sum involving binomial coefficients, Fibonacci, Lucas, and Bernoulli numbers

H-852 Proposed by Robert Frontczak, Stuttgart, Germany
(Vol. 58, No. 1, February 2020)

Let $(B_n)_{n\geq 0}$ denote the Bernoulli numbers. Show that for all $r \geq 1$ and $n \geq 3$,

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} L_{r(n-k)} B_k B_{n-k} = \begin{cases} (1-n)B_n F_{rn}, & n \text{ even;} \\ -nB_{n-1}F_{rn}, & n \text{ odd.} \end{cases}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (2^{1-k}-1)(2^{1-(n-k)}-1) F_{rk} L_{r(n-k)} B_k B_{n-k} = \begin{cases} (1-n)B_n F_{rn}, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases}$$

Solution by the proposer

From [2] we know that if a sequence of numbers $T(n,k)$ satisfies the relation

$$T(n,k) = T(n,n-k) \quad (0 \leq k \leq n),$$

then

$$\sum_{k=0}^{n} T(n,k) F_{rk} L_{r(n-k)} = F_{rk} \sum_{k=0}^{n} T(n,k).$$

We apply this relation to the Bernoulli polynomials, which are defined by

$$\frac{ze^{xz}}{e^z-1} = \sum_{n\geq 0} B_n(x) \frac{z^n}{n!} \quad (|z| < 2\pi).$$

Recall that, for $n \geq 1$, we have the following relation for the Bernoulli polynomials (see [1]):

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) = n(x+y-1)B_{n-1}(x+y) - (n-1)B_n(x+y).$$

So, setting $x = y$ we get the special case

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(x) = n(2x-1)B_{n-1}(2x) - (n-1)B_n(2x).$$

Now, by applying Carlitz's formula with

$$T(n,k) = \binom{n}{k} B_k(x) B_{n-k}(x),$$

we get the more general statement

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} L_{r(n-k)} B_k(x) B_{n-k}(x) = F_{rn}(n(2x-1)B_{n-1}(2x) - (n-1)B_n(2x)).$$

For $x = 0$, and noting that $B_n(0) = B_n$, we get

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} L_{r(n-k)} B_k B_{n-k} = F_{rn}(-nB_{n-1} - (n-1)B_n).$$

The first identity follows because $B_{2n+1} = 0$ for $n \geq 1$. The second identity is a special case when $x = 1/2$, using $B_n(1/2) = (2^{1-n}-1)B_n$, $B_n(1) = (-1)^n B_n$, and simplifying.

References

[1] T. Agoh, *Convolution identities for Bernoulli and Genocchi polynomials*, The Electronic Journal of Combinatorics, **21.1** (2014), #P1.65.
[2] L. Carlitz, *Solution to Problem H-285*, The Fibonacci Quarterly, **18.2** (1980), 191–192.

Also solved by Brian Bradie, Dmitry Fleischman, G. C. Greubel, Raphael Schumacher, and Albert Stadler.

Lower bounds for some sums involving Lucas numbers

H-853 Proposed by Ángel Plaza and Sergio Falcón, Gran Canaria, Spain (Vol. 58, No. 1, February 2020)

Let L_n be the nth k-Lucas number given by the recurrence $L_{n+2} = kL_{n+1} + L_n$ for all $n \geq 0$, with $L_0 = 2$, $L_1 = k$. Prove that

(i) $\displaystyle\sum_{j=1}^{n} \frac{L_j^2}{\sqrt{L_j+1}} \geq \frac{(L_n + L_{n+1} - k - 2)^2}{k\sqrt{kn(L_n + L_{n+1} + k(n-1) - 2)}}$;

(ii) $\displaystyle\sum_{j=1}^{n} \frac{L_j^4}{\sqrt{L_j^2+1}} \geq \frac{(L_{2n+1} + k((-1)^n - 2))^2}{k\sqrt{kn(L_{2n+1} + k(n-2+(-1)^n))}}$.

Solution by the proposers

The inequalities follow by Jensen's inequality. Note that the function $f(x) = \dfrac{x^2}{\sqrt{x+1}}$ is convex because $f''(x) = \dfrac{3x^2+8x+8}{4(x+1)^{5/2}} > 0$. Therefore,

$$\begin{aligned}\sum_{j=1}^{n} \frac{L_j^2}{\sqrt{L_j+1}} &\geq n \cdot \frac{\left(\frac{\sum L_j}{n}\right)^2}{\sqrt{\frac{\sum L_j}{n}+1}} \\ &= n \cdot \frac{\left(\frac{L_n+L_{n+1}-k-2}{kn}\right)^2}{\sqrt{\frac{L_n+L_{n+1}-k-2}{kn}+1}} = \frac{(L_n + L_{n+1} - k - 2)^2}{k\sqrt{kn(L_n + L_{n+1} + k(n-1) - 2)}},\end{aligned}$$

where we use $\displaystyle\sum_{j=1}^{n} L_j = \frac{L_n + L_{n+1} - k - 2}{k}$, which can be proved by induction or by using the Binet's formula for k-Lucas numbers.

Inequality (ii) follows by Jensen's inequality as before, and using that $\displaystyle\sum_{j=1}^{n} L_j^2 = \frac{L_{2n+1}}{k} + (-1)^n - 2$, which may be proved by induction or by using the Binet's formula for k-Lucas numbers:

$$\begin{aligned}\sum_{j=1}^{n} \frac{L_j^4}{\sqrt{L_j^2+1}} &\geq n \cdot \frac{\left(\frac{\sum L_j^2}{n}\right)^2}{\sqrt{\frac{\sum L_j^2}{n}+1}} \\ &= n \cdot \frac{\left(\frac{L_{2n+1}+(-1)^n k-2k}{kn}\right)^2}{\sqrt{\frac{L_{2n+1}+(-1)^n k-2k}{kn}+1}} = \frac{\left(L_{2n+1}+k\left((-1)^n-2\right)\right)^2}{k\sqrt{kn\left(L_{2n+1}+k\left(n-2+(-1)^n\right)\right)}}.\end{aligned}$$

□

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, G. C. Greubel, and Albert Stadler.