ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-755 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Let $n \geq 1$ be an integer. Prove that

(1) If $x_k \in \mathbb{R}$ for $k = 1, \ldots, n$, then

$$2\left(\sum_{k=1}^{n} L_k \sin x_k\right)\left(\sum_{k=1}^{n} L_k \cos x_k\right) \leq n(L_n L_{n+1} - 2).$$

(2) If $m \geq 1$, then

$$m^m \sum_{k=1}^{n} (1 + L_{2k-1})^{m+1} \geq (m+1)^{m+1}(L_{2n+2} - 2).$$

H-756 Proposed by Russell J. Hendel, Towson University.

We seek to generalize a known problem which states that

$$\#\{\langle x_1, \ldots, x_{n+1}\rangle : x_1x_2 \vee x_2x_3 \vee \cdots \vee x_nx_{n+1} = 0\} = F_{n+3},$$

where x_i are Boolean variables for $i = 1, \ldots, n$. To generalize the above formula, we

(i) fix integers $d,\ i$ with $d > i \geq 1$;
(ii) let D_j be products of d Boolean variables x_k with consecutive indices such that D_j and D_{j+1} have i variables in common;
(iii) let m be the total number of variables occurring in $D_1, \ldots, D_n$ and
(iv) let

$$S_n = \#\{\langle x_1, \ldots, x_m\rangle : D_1 \vee D_2 \vee \cdots \vee D_n = 0\}.$$

Determine the coefficients of the minimal recursion satisfied by the $\{S_n\}_{n\geq 1}$.

H-757 Proposed by H. Ohtsuka, Saitama, Japan.

For an odd prime p prove that

$$\prod_{k=1}^{p} L_k \equiv \begin{cases} 2(-1)^{(p+1)/4} & \pmod{F_p} \quad \text{if} \quad p \equiv -1 \pmod 4, \\ (-1)^{(p-1)/4} F_{p-3} & \pmod{F_p} \quad \text{if} \quad p \equiv 1 \pmod 4. \end{cases}$$

H-758 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Compute:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}^{F_m}\left(\sqrt[n]{(2n-1)!!}^{F_{m+1}}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right)^{F_{m+2}}\right)\right).$$

H-759 Proposed by H. Ohtsuka, Saitama, Japan.

Let $r \geq 2$ be an integer. Define the sequence $\{G_n\}$ by

$$G_n = G_{n-1} + \cdots + G_{n-r} \qquad (n \geq 1)$$

with arbitrary $G_0, G_1, \ldots, G_{-r+1}$. For an integer $n \geq 1$, prove that

$$\sum_{k=1}^{n} G_k^2 = \sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} \sum_{i=1}^{k} (G_{n+i-k}G_{n+i} - G_{i-k}G_i).$$

SOLUTIONS

An Identity Involving Middle Binomial Coefficients

H-727 Proposed by Bassem Ghalayini, Louaize, Lebanon.
(Vol. 50, No. 3, August 2012)

Let n be a natural number. Prove that

$$(2n+1)\binom{2n}{n} = \sum_{\substack{0\leq i,j,k\leq n \\ i+j+k=n}} \binom{2i}{i}\binom{2j}{j}\binom{2k}{k}.$$

Solution by Eduardo Brietzke.

The generating function for the central binomial coefficients is

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n. \tag{1}$$

Replacing x by x^2 and multiplying by x we obtain

$$\frac{x}{\sqrt{1-4x^2}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^{2n+1}.$$

Differentiating we get

$$\frac{1}{(1-4x^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} (2n+1)\binom{2n}{n} x^{2n}.$$

Replacing back x^2 by x and using (1) yields

$$\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right)^3 = \sum_{n=0}^{\infty}(2n+1)\binom{2n}{n}x^n,$$

or,

$$\sum_{n=0}^{\infty}x^n \sum_{\substack{0\le i,j,k\le n\\ i+j+k=n}}\binom{2i}{i}\binom{2j}{j}\binom{2k}{k} = \sum_{n=0}^{\infty}(2n+1)\binom{2n}{n}x^n,$$

and the desired identity follows.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Ángel Plaza, and the proposer.

Inequalities With Consecutive Fibonacci Numbers, Square–Roots and Powers

H-728 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania. (Vol. 50, No. 4, November 2012)

Let a, b, c, m be positive real numbers and n be a positive real number. Prove that:

(a) $$\frac{F_n}{\sqrt{F_n^2 + aF_{n+1}F_{n+2}}} + \frac{F_{n+1}}{\sqrt{F_{n+1}^2 + bF_{n+2}F_n}} + \frac{F_{n+2}}{\sqrt{F_{n+2}^2 + cF_nF_{n+1}}} \ge 1,$$

provided that $a + b + c \le 24$;

(b) $$\frac{a^{-3m-3}}{(F_nb + F_{n+1}c)^{m+1}} + \frac{b^{-3m-3}}{(F_nc + F_{n+1}a)^{m+1}} + \frac{c^{-3m-3}}{(F_na + F_{n+1}b)^{m+1}} \ge \frac{3}{F_{n+2}^{m+1}},$$

provided that $abc = 1$.

Solution by Ángel Plaza.

Part (a) is a direct consequence of a more general inequality: Let x, y, z, a, b, c be positive real numbers, with $a + b + c \le 24$. Then

$$\frac{x}{\sqrt{x^2 + ayz}} + \frac{y}{\sqrt{y^2 + bzx}} + \frac{z}{\sqrt{z^2 + cxy}} \ge 1.$$

By Hölder's inequality

$$\left(\sum \frac{x}{\sqrt{x^2 + \mu yz}}\right)\left(\sum \frac{x}{\sqrt{x^2 + \mu yz}}\right)\left(\sum x(x^2 + \mu yz)\right) \ge (x + y + z)^3,$$

where the sums are cyclic and the coefficient μ means the corresponding coefficient a, b, or c. Therefore, we need only to show that

$$(x + y + z)^3 \le x^3 + y^3 + z^3 + (a + b + c)xyz,$$

which is equivalent to

$$(x + y)(y + z)(z + x) \ge \frac{(a + b + c)}{3}xyz.$$

And this is true due to the AM-GM inequality:

$$\left(\frac{x + y}{\sqrt{xy}}\right)\left(\frac{y + z}{\sqrt{yz}}\right)\left(\frac{z + x}{\sqrt{zx}}\right) \ge 8 \ge \frac{a + b + c}{3}.$$

□

For part (b) it is enough to prove the following more general inequality: Let x, y, a, b, c be positive real numbers, with $abc = 1$. Then

$$\frac{1}{a^3(xb+yc)} + \frac{1}{b^3(xc+ya)} + \frac{1}{c^3(xa+yb)} \geq \frac{3}{x+y},$$

which is a consequence of

$$(xb+yc)(xc+ya)(xa+yb) \leq (x+y)^3,$$

since by the AM-GM inequality $a^2b + b^2c + c^2a \leq 3$. □

Also solved by Paul S. Bruckman, Dmitry Fleischman, and the proposers.

On the Sequence of Rotational Numbers

H-729 Proposed by Paul S. Bruckman, Nanaimo, BC.
(Vol. 50, No. 4, November 2012)

Define a sequence $\{a_n\}_{n\geq 0}$ of rational numbers by the recurrence $\sum_{k=0}^{n} \frac{a_k}{n+1-k} = \delta_{n,0}$, where $\delta_{i,j}$ is the Kronecker symbol which equals 1 if $i = j$ and 0; otherwise.

(a) Prove that $-\sum_{k=1}^{\infty} \frac{a_n}{n} = \gamma$, the Euler constant;

(b) Prove that $a_n = -\frac{1}{n+1} + \sum_{k=0}^{n-1} u_{n-k} a_k$ for $n \geq 1$, where $u_m = \frac{2(H_m - 1)}{(m+2)}$ and $H_m = \sum_{k=1}^{m} \frac{1}{k}$ for all $m \geq 1$.

Solution by Anastasios Kotronis.

We start proving a lemma.

Lemma 1. *For $n \geq 0$, $a_n \geq -1$.*

Proof. From the recurrence we get $a_0 = 1 \geq -1$ and $a_1 = -1/2 \geq -1$. Assume that $a_m \geq -1$ for $1 \leq m \leq n$. From the recurrence, we get

$$(n+1)a_n + \frac{n+1}{2}a_{n-1} + \frac{n+1}{3}a_{n-2} + \cdots + \frac{n+1}{n}a_1 + 1 = 0, \tag{1}$$

$$(n+2)a_{n+1} + \frac{n+2}{2}a_n + \frac{n+2}{3}a_{n-1} + \cdots + \frac{n+2}{n+1}a_1 + 1 = 0. \tag{2}$$

Now using the induction hypothesis and subtracting (1) from (2), we get:

$$\begin{aligned} a_{n+1} &= \frac{1}{n+2}\sum_{k=1}^{n} a_k \left(\frac{n+1}{n+1-k} - \frac{n+2}{n+2-k}\right) \\ &\geq \frac{1}{n+2}\sum_{k=1}^{n} \left(\frac{n+2}{n+2-k} - \frac{n+1}{n+1-k}\right) \\ &= H_n\left(1 - \frac{n+1}{n+2}\right) + \frac{1}{n+1} - 1 = \frac{H_n}{n+2} + \frac{1}{n+1} - 1 \geq -1. \end{aligned}$$

□

Next, we compute the generating function of a_n. Multiplying by x^n and summing for the recurrence defining a_n for $n \geq 0$, we get

$$\sum_{n\geq 0}\sum_{k=0}^{n} \frac{a_k}{n+1-k} x^n = \sum_{n\geq 0} \delta_{n,0} x^n,$$

or, equivalently,

$$\sum_{k\geq 0} a_k x^k \sum_{k\geq 0} \frac{x^k}{k+1} = 1,$$

or, equivalently,

$$-\frac{\ln(1-x)}{x} \sum_{k\geq 0} a_k x^k = 1,$$

therefore,

$$A(x) := \sum_{k\geq 0} a_k x^k = -\frac{x}{\ln(1-x)},$$

from where we get that $\sum_{n\geq 0} a_n x^n$ has radius of convergence 1.

Now for $x \in (-1, 1)$, we have

$$A(x) = 1 + x \sum_{n\geq 1} a_n x^{n-1},$$

so,

$$\frac{A(x)-1}{x} = \sum_{n\geq 1} a_n x^{n-1},$$

which implies that

$$\int_0^x \frac{A(t)-1}{t}\, dt = \sum_{n\geq 1} \frac{a_n}{n} x^n.$$

But from Lemma 1, we have that $a_n/n \geq -1/n$ so from a known theorem of Hardy and Littlewood (see [1, p. 65, Theorem 8.4]), we have that

$$\sum_{n\geq 1} \frac{a_n}{n}$$

converges and

$$-\sum_{n\geq 1} \frac{a_n}{n} = -\int_0^1 \frac{A(t)-1}{t}\, dt = \int_0^1 \frac{1}{\ln t} + \frac{1}{t}\, dt = \gamma$$

(see [2, p. 1]). This proves part (a).

(b) With the convention that the sum over an empty set of indices is zero, we get that $H_0 = 0$, so $u_0 = -1$ and what we want to prove is equivalent to

$$\sum_{k=0}^{n} a_k u_{n-k} = \frac{1}{n+1}, \qquad (n \geq 1).$$

Denoting by $U(x)$ the generating function of u_n, multiplying by x^n and summing for $n \geq 1$, we get

$$\sum_{n\geq 1}\sum_{k=0}^{n} a_k u_{n-k} x^n = \sum_{n\geq 1} \frac{x^n}{n+1},$$

therefore,

$$A(x)U(x) - a_0u_0 = (A(x))^{-1} - 1,$$

so

$$U(x) = \frac{\ln^2(1-x)}{x^2} + 2\frac{\ln(1-x)}{x},$$

and it suffices to show that

$$[x^n]\left(\frac{\ln^2(1-x)}{x^2} + 2\frac{\ln(1-x)}{x}\right) = \frac{2(H_n-1)}{n+2},$$

where $[x^n]f(x)$ denotes the coefficient of x^n in the Taylor expansion of $f(x)$. But

$$\begin{aligned}[x^n]\frac{\ln^2(1-x)}{x^2} &= \sum_{k=0}^{n}\frac{1}{(k+1)(n-k+1)}\\ &= -\frac{1}{n+2}\sum_{k=0}^{n}\left(\frac{1}{k+1}+\frac{1}{n-k+1}\right)\\ &= \frac{2H_{n+1}}{n+2},\end{aligned}$$

so

$$\begin{aligned}[x^n]\left(\frac{\ln^2(1-x)}{x^2} + 2\frac{\ln(1-x)}{x}\right) &= \frac{2H_{n+1}}{n+2} - \frac{2}{n+1}\\ &= \frac{2\left(H_n+\frac{1}{n+1}\right)}{n+2} - \frac{2}{n+1}\\ &= \frac{2H_n}{n+2} + \frac{2}{(n+1)(n+2)} - \frac{2}{n+1}\\ &= \frac{2(H_n-1)}{n+2}.\end{aligned}$$

References

[1] A. M. Odlyzko, *Asymptotic enumeration methods*,
`http://www.dtc.umn.edu/~odlyzko/doc/asymptotic.enum.pdf`.

[2] P. Sebah and X. Gourdon, *Collection of formulae for Euler's constant* γ,
`http://numbers.computation.free.fr/Constants/Gamma/gammaFormulas.pdf`.

Also solved by G. C. Greubel and the proposer.

Identities Involving Fibonacci, Lucas and Pell Numbers

H-730 Proposed by N. Gauthier, Kingston, ON.
(Vol. 51, No. 1, February 2013)

Let $\lfloor x\rfloor$ be the largest integer less than or equal to x and put $\varepsilon_n = (1+(-1)^n)/2$. Then, with P_n the nth Pell number prove the following identities:

(a) $\displaystyle\sum_{k\ge 0}\frac{1}{25^k}\binom{n-2k}{2k} = \frac{1}{5^{n/2}6}\left[\varepsilon_n(L_{2n+2}+3L_{n+1}) + (1-\varepsilon_n)\sqrt{5}(F_{2n+2}+3F_{n+1})\right];$

(b) $\displaystyle\sum_{k\ge 0}\frac{1}{16^k}\binom{n-1-2k}{2k} = \frac{1}{2^n}[P_n+n];$

(c)

$$\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \frac{1}{25^k(n-4k)}\binom{n-1-2k}{2k} = \frac{1}{5^{n/2}n}\Big[\varepsilon_n(L_{2n}+L_n-2(1+(-1)^{n/2})) + (1-\varepsilon_n)\sqrt{5}(F_{2n}+F_n)\Big];$$

(d)

$$\sum_{k\geq 1} \frac{k}{5^k}\binom{n-1-k}{k} = \frac{1}{5^{n/2}54}\Big[\varepsilon_n((45n-20)F_{2n}-15nL_{2n}) + (1-\varepsilon_n)\sqrt{5}((9n-4)L_{2n}-15nF_{2n})\Big].$$

Solution by Ángel Plaza.

(a) In order to prove the equality we will show that both sides of the equality present the same generating function. For the left-hand side we use the "Snake Oil Method" [1] applied to

$$a_n = \sum_{k\geq 0} \frac{1}{25^k}\binom{n-2k}{2k}.$$

Let $A(x)$ be the generating function of $\{a_n\}$. That is,

$$\begin{aligned} A(x) &= \sum_{n\geq 0} x^n \sum_{k\geq 0} \frac{1}{25^k}\binom{n-2k}{2k} = \sum_{k\geq 0} \frac{x^{2k}}{25^k} \sum_{n\geq 0} \binom{n-2k}{2k} x^{n-2k} \\ &= \sum_{k\geq 0} \frac{x^{2k}}{25^k} \frac{x^{2k}}{(1-x)^{2k+1}} = \frac{1}{1-x}\sum_{k\geq 0}\left(\frac{x^4}{25(1-x)^2}\right)^k \\ &= \frac{25(1-x)}{25(1-x)^2-x^4}, \end{aligned}$$

where we have used the identity

$$\sum_{r\geq 0} \binom{rk}{x}^r = \frac{x^k}{\sqrt{(1-x)^{k+1}}} \qquad (k\geq 0),$$

(see eq. (4.3.1), page 120 in [1]). Then, the generating function for the even terms of the considered sequence is

$$A_e(x) = \frac{A(x)+A(-x)}{2} = \frac{-25(-25+25x^2+x^4)}{625-1250x^2+575x^4-50x^6+x^8},$$

and the generating function for the odd terms of the sequence is

$$A_o(x) = \frac{A(x)-A(-x)}{2} = \frac{25x(25-25x^2+x^4)}{625-1250x^2+575x^4-50x^6+x^8}.$$

For the right-hand side of (a) we consider the two cases: n even, and n odd, since respectively $\varepsilon_n = 1$ and $\varepsilon_n = 0$. By using the Binet's formulas for Lucas and Fibonacci numbers and the sum of geometric series it is a routine to get the same generating functions obtained for the left-hand side of (a), $A_e(x)$ and $A_o(x)$.

(b) Let $A(x)$ be the generating function of the sequence at the left-hand side. That is,

$$\begin{aligned}
A(x) &= \sum_{n\geq 0} x^n \sum_{k\geq 0} \frac{1}{16^k}\binom{n-1-2k}{2k} = \sum_{k\geq 0} \frac{x^{2k+1}}{16^k} \sum_{n\geq 0} \binom{n-1-2k}{2k} x^{n-1-2k} \\
&= x\sum_{k\geq 0}\left(\frac{x^2}{16}\right)^k \frac{x^{2k}}{(1-x)^{2k+1}} = \frac{x}{1-x}\sum_{k\geq 0}\left(\frac{x^4}{16(1-x)^2}\right)^k \\
&= \frac{16x(1-x)}{16(1-x)^2 - x^4}.
\end{aligned}$$

For the right-hand side of (b) we may use the Binet's formula for the Pell numbers, or more directly its generating function $P(x) = \dfrac{x}{1-2x-x^2}$. Then

$$\sum_{n\geq 0} P_n \frac{x^n}{2^n} = \frac{x/2}{1-2(x/2)-(x/2)^2} = \frac{2x}{4-x-x^2}.$$

Also, since

$$\sum_{n\geq 0} \frac{1}{2^n} x^n = \frac{2}{2-x},$$

we then have

$$\sum_{n\geq 0} \frac{n}{2^n} x^n = x\frac{d}{dx}\left(\frac{2}{2-x}\right) = \frac{2x}{(2-x)^2}.$$

Finally, since

$$\frac{2x}{4-x-x^2} + \frac{2x}{(2-x)^2} = \frac{16x(1-x)}{16(1-x)^2-x^4},$$

the result follows.

Identities (c) and (d) may be proved by similar arguments to the used for identity (a).

References

[1] H. S. Wilf, *Generatingfunctionology*, Academic Press, Inc., Second ed. 1994.

Also solved by Paul S. Bruckman, G. C. Greubel, and the proposer.