ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-677 Proposed by N. Gauthier, Kingston, ON, Canada

Let $N \geq 3$ be an integer and define $Q = \lfloor (N-1)/2 \rfloor$. Find a closed form expression for the following sum

$$S(N) = \sum_{k=1}^{Q} \frac{k \sin((2k+1)\pi/N)}{\sin^2(k\pi/N)\sin^2((k+1)\pi/N)}.$$

H-678 Proposed by Mohammad K. Azarian, Evansville, IN

(a) Show that there is a unique Fibonacci number F such that the inequalities

$$x_1 + x_2 + \cdots + x_{70} < F \qquad \text{and} \qquad y_1 + y_2 + \cdots + y_{18} < F$$

have the same number of positive integer solutions.

(b) Show that it is impossible to find three consecutive Fibonacci numbers $F_k,\ F_{k+1},\ F_{k+2}$ such that the inequalities

$$x_1 + x_2 + \cdots + x_{F_k} < F_{k+2} \qquad \text{and} \qquad y_1 + y_2 + \cdots + y_{F_{k+1}} < F_{k+2}$$

have the same number of positive integer solutions.

H-679 Proposed by N. Gauthier, Kingston, ON, Canada

For integers $a \geq 1$ and $n \geq 0$ consider the generalized Fibonacci sequence $\{f_n\}_n$ given by $f_0 = 0,\ f_1 = 1$ and $f_{n+2} = af_{n+1} + f_n$ for $n \geq 0$. Let $\Delta = \sqrt{a^2+4}$ and $\alpha = (a+\Delta)/2,\ \beta = (a-\Delta)/2$ be the roots of the characteristic equation of the recurrence. Consider the sequence $\{S_n\}_{n\geq 4}$ of nested radical sums

$$A_n = \sqrt{f_4 + \sqrt{f_5 + \cdots + \sqrt{f_n}}}.$$

Prove that

$$S_n < \frac{\alpha^{6+p(n)}}{\Delta^{1+q(n)}},$$

where $p(n)$ and $q(n)$ are to be determined, and find an upper bound for the limit $S = \lim_{n\to\infty} S_n$.

H-680 Proposed by N. Gauthier, Kingston, ON, Canada

For $x \neq 0$ an indeterminate and for an integer $n \geq 0$, consider the generalized Fibonacci and Lucas polynomials $\{f_n\}_n$ and $\{l_n\}_n$, respectively, given by the following recurrences

$$\begin{aligned} f_{n+2} &= xf_{n+1} + f_n \qquad n \geq 0, \quad \text{where} \quad f_0 = 0,\ f_1 = 1; \\ l_{n+2} &= xl_{n+1} + l_n \qquad n \geq 0, \quad \text{where} \quad l_0 = 2,\ l_1 = x. \end{aligned}$$

Find closed-form expressions for the following sums:

(a) $\displaystyle\sum_{k=1}^{m}(-1)^{kn}\frac{1}{f_{(k+1)n}f_{kn}}, \quad m,\ n \geq 1;$

(b) $\displaystyle\sum_{k=0}^{m}(-1)^{kn}\frac{1}{l_{(k+1)n}l_{kn}}, \quad m,\ n \geq 0;$

(c) $\displaystyle\sum_{k=1}^{m}(-1)^{kn}\frac{f_{(2k+1)n}}{f^2_{(k+1)n}f^2_{kn}}, \quad m,\ n \geq 1;$

(d) $\displaystyle\sum_{k=0}^{m}(-1)^{kn}\frac{f_{(2k+1)n}}{l^2_{(k+1)n}l^2_{kn}}, \quad m,\ n \geq 0;$

(e) $\displaystyle\sum_{k=0}^{m}(-1)^{kn}\frac{f_{(2k+1)n}[f^2_{(2k+1)n} + f^2_n]}{l^4_{(k+1)n}l^4_{kn}}, \quad m,\ n \geq 0.$

SOLUTIONS

Combinatorial Numbers and Powers of an Arithmetic Progression

H-659 Proposed by N. Gauthier, Kingston, ON, Canada (Vol. 45, No. 3, August 2007)

Let $m \geq 0,\ n \geq 0,\ a,\ b,\ q$ and s be integers. Also let $\{c(m,k) = c(m,k;n,a,b) : 0 \leq k \leq m\}$ represent the set of coefficients that are given by the following recurrence, with the initial value and boundary conditions shown:

$$\begin{aligned} &c(m+1,k) = (b+ka)c(m,k) + (n-k+1)ac(m,k-1); \\ &c(0,0) = 1;\ c(m,-1) = c(m,m+1) = 0. \end{aligned}$$

Also let $F_{n+2} = uF_{n+1} + F_n$ and $L_{n+2} = uL_{n+1} + L_n$ be the recurrences for the generalized Fibonacci and Lucas polynomials of order n, $F_n = F_n(u)$ and $L_n = L_n(u)$, respectively, where $F_0 = 0,\ F_1 = 1$, and $L_0 = 2,\ L_1 = u$. Prove that the following identities hold:

a) $\sum_{r=0}^{n}\binom{n}{r}(ar+b)^m = \sum_{k=0}^{m} 2^{n-k}c(m,k)$;

b) $\sum_{r=0}^{n}(-1)^{qa(n-r)}\binom{n}{r}(ar+b)^m F_{2qar+s} = \sum_{k=0}^{m} c(m,k)L_{qa}^{n-k}F_{qa(n+k)+s}$.

Solution by the proposer

Consider the following function of the variable x:

$$S_0(x;n,a,b) = x^b(1+x^a)^n = \sum_{r=0}^{n}\binom{n}{r}x^{ar+b}. \tag{1}$$

The right-hand side above follows from a binomial expansion. For simplicity, let $S_0(x)$ stand for $S_0(x; n, a, b)$ and consider the repeated action of the differential operator, $D = x\dfrac{d}{dx}$, on that function. From (1), we see that for a nonnegative integer m one has

$$S_m(x) = D^m S_0(x) = \sum_{r=0}^{n} \binom{n}{r} (ar+b)^m x^{ar+b}. \tag{2}$$

Also, again from (1), one has that

$$S_m(x) = D^m[x^b(1+x^a)^n]. \tag{3}$$

To develop this latter expression, consider a few terms to see a pattern.

$$\begin{aligned} S_1(x) &= D[x^b(1+x^a)^n] = bx^b(1+x^a)^n + nax^{a+b}(1+x^a)^{n-1}; \\ S_2(x) &= D^2 S_0(x) = DS_1(x) \\ &= b^2x^b(1+x^a)^n + an(a+2b)x^{a+b}(1+x^a)^{n-1} + n(n-1)a^2x^{2a+b}(1+x^a)^{n-2}. \end{aligned}$$

For an arbitrary nonnegative integer m we have that

$$S_m(x) = \sum_{k=0}^{m} c(m,k)x^{ka+b}(1+x^a)^{n-k}. \tag{4}$$

We now determine a recurrence for the unknown coefficients $\{c(m,k)\}$, as follows. First replace m by $m+1$ in (4) and get that

$$S_{m+1}(x) = \sum_{k=0}^{m+1} c(m+1,k)x^{ka+b}(1+x^a)^{n-k}. \tag{5}$$

Also, by (3) and (4), we have that

$$\begin{aligned} S_{m+1}(x) &= DS_m(x) \\ &= D\sum_{k=0}^{m} c(m,k)x^{ka+b}(1+x^a)^{n-k} \\ &= \sum_{k=0}^{m} c(m,k)[(ka+b)x^{ka+b}(1+x^a)^{n-k} + (n-k)ax^{(k+1)a+b}(1+x^a)^{n-k-1}] \\ &= \sum_{k=0}^{m+1}[(ka+b)c(m,k) + (n-k+1)ac(m,k-1)]x^{ka+b}(1+x^a)^{n-k}. \end{aligned} \tag{6}$$

The definitions $c(m,-1) = 0$ and $c(m,m+1) = 0$ make it possible to readjust the limits in the sums in the third line of (6) above as shown in the fourth line of (6) above. The recurrence for the coefficients we seek is thus given by equating the coefficient of $x^{ka+b}(1+x^a)^{n-k}$ in the right-hand of (5) to that of the same term in the last line of (6). The resulting recurrence is as given in the problem statement.

To get identity a), set $x = 1$ in the right-hand side of (2) and in the right-hand side of (4). Then equate the results to get identity a).

To get identity b), set $x = \alpha^q/\beta^q = (-1)^q\alpha^{2q}$, where $\alpha = \alpha(u) = (u+\sqrt{u^2+1})/2$ and $\beta = \beta(u) = (u-\sqrt{u^2+1})/2$ are the roots of the characteristic equation for the generalized Fibonacci and Lucas polynomials; namely, the numbers w such that $w^2 - uw - 1 = 0$.

Inserting the given substitution in (2) and (4) and then equating the above results gives, upon multiplication of the entire equation by α^p, with p an integer, that

$$\begin{aligned}\sum_{r=0}^{n}\binom{n}{r}(ar+b)^m(-\alpha^2)^{q(ar+b)}\alpha^p &= \sum_{k=0}^{m}c(m,k)(-\alpha^2)^{q(ka+b)}(1+\alpha^{qa}/\beta^{qa})^{n-k}\alpha^p\\ &= \sum_{k=0}^{m}c(m,k)(-\alpha^2)^{q(ka+b)}L_{qa}^{n-k}(-\alpha)^{qa(n-k)}\alpha^p\\ &= \sum_{k=0}^{m}c(m,k)(-1)^{q(na+b)}L_{qa}^{n-k}\alpha^{q(ka+2b+na)+p}.\end{aligned}$$

Next change α to β in this expression, then subtract the resulting expression from the one above and divide both sides of the resulting equation by $(\alpha-\beta)$. Identity b) follows upon eliminating a redundant sign, $(-1)^{qb}$, on both sides of the resulting equality, then transferring $(-1)^{qna}$ to the left-hand side and setting $s := 2qb+p$.

Also solved by Paul Bruckman and G. C. Greubel.

On Odd Perfect Numbers

H-661 Proposed by J. López González, Madrid, Spain and F. Luca, Mexico (Vol. 45, No. 4, November 2007)

Let $\phi(n)$ and $\sigma(n)$ be the Euler function of n and the sum of divisors function of n, respectively.

(i) If n is odd perfect show that $0.4601 < \phi(n)/n < 0.5$.
(ii) Show that n is odd perfect if and only if $n\sigma(2n) = \sigma(n)(n+\sigma(n))$.

Solution by the proposers

(i) First of all, it is known that

$$1 > \frac{\phi(n)\sigma(n)}{n^2} > \frac{6}{\pi^2}. \tag{7}$$

Let us recall a proof of it. If $n = p^a$ is a prime power then

$$\frac{\phi(p^a)\sigma(p^a)}{p^{2a}} = \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\cdots+\frac{1}{p^a}\right).$$

The above expression is at least

$$\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right) = 1-\frac{1}{p^2} \quad \text{and less than} \quad \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = 1.$$

Since $\phi(n), \sigma(n)$ and n are all three multiplicative, so is $\phi(n)\sigma(n)/n^2$, so

$$1 > \frac{\phi(n)\sigma(n)}{n^2} \ge \prod_{p\,|\,n}\left(1-\frac{1}{p^2}\right) > \prod_{p\ge 2}\left(1-\frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Since $\sigma(n) = 2n$, the claimed upper bound on $\phi(n)/n$ follows. To get the lower bound, we revisit the proof of the lower bound above on $\phi(n)\sigma(n)/n^2$ using more information about the arithmetic of the odd perfect number n. Since n is odd, the prime $p = 2$ does not divide n. So, in fact, we see that

$$\frac{\phi(n)\sigma(n)}{n^2} \geq \prod_{p\geq 3}\left(1-\frac{1}{p^2}\right) = \frac{4}{3\zeta(2)} = \frac{8}{\pi^2}.$$

Still this does not take full advantage of the structure of odd perfect numbers. In fact, the last bound only used the fact that n is odd and nothing else. A result due to Euler says that if n is odd perfect then $n = qm^2$, where q is prime with $q \equiv 1 \pmod 4$. In conclusion, for all prime factors $p \neq q$ of n, we have $p^a \mid n$ for some $a \geq 2$, and so

$$\frac{\phi(p^a)\sigma(p^a)}{p^{2a}} = \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\cdots+\frac{1}{p^a}\right) \geq \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\frac{1}{p^2}\right) \geq \left(1-\frac{1}{p^3}\right).$$

Thus, if all primes p dividing n appear at a power ≥ 2, we then have, again by multiplicativity,

$$\frac{\phi(n)\sigma(n)}{n^2} \geq \prod_{p\geq 3}\left(1-\frac{1}{p^3}\right) = \frac{8}{7\zeta(3)}.$$

However, this is not quite so since the prime q might happen to exactly divide n; i.e., $q\|n$, but by Euler's theorem this prime is unique and congruent to $1 \pmod 4$. The first such possible prime is 5. Thus, if we replace the contribution of the prime $p = 5$ above (which is $1 - 1/125 = 124/125$) by the (smaller) amount $1 - 1/25 = 24/25$, then we get the inequality

$$\frac{\phi(n)\sigma(n)}{n^2} \geq \frac{8}{7\zeta(3)}\,\frac{125}{124}\,\frac{24}{25} = \frac{240}{217\zeta(3)},$$

valid for all odd perfect numbers n. Since $\zeta(3) = 1.2020569\ldots$, we get that

$$\frac{240}{217\zeta(3)} = .92008\ldots.$$

Since $\sigma(n) = 2n$, we get that

$$\frac{\phi(n)}{n} > \frac{120}{217\zeta(3)} = .46004\ldots.$$

(ii) Write $n = 2^a m$, where $a \geq 0$ and m is odd. Then the given equation implies

$$2^a m(2^{a+2}-1)\sigma(m) = (2^{a+1}-1)\sigma(m)(2^a m + (2^{a+1}-1)\sigma(m)),$$

which after simplification by $\sigma(m)$ becomes

$$2^a(2^{a+2}-1)m = 2^a(2^{a+1}-1)m + (2^{a+1}-1)^2\sigma(m).$$

This gives

$$(2^{a+1}-1)^2\sigma(m) = m\left[2^a(2^{a+2}-1) - 2^a(2^{a+1}-1)\right]m = 2^{2a+1}m.$$

Thus,

$$\sigma(m) = \frac{2^{2a+1}}{(2^{a+1}-1)^2}m. \tag{8}$$

One checks easily that if $a \geq 1$, then $(2^{a+1}-1)^2 = 2^{2a+2} - 2^{a+2} + 1 > 2^{2a+1}$, since this is equivalent to $2^{2a+2} - 2^{2a+1} > 2^{a+2} - 1$, or $2^{2a+1} > 2^{a+2} - 1$, which is implied by $2a+1 \geq a+2$, which in turn is true since $a \geq 1$. But in this case the factor $2^{2a+1}/(2^{a+1}-1)^2$ appearing in front of m in the right hand side of (8) is < 1, leading to the conclusion that $\sigma(m) < m$, which is certainly false. So the only chance is that $a = 0$, in which case $2^{2a+1}/(2^{a+1}-1)^2 = 2$ and the relation (8) becomes $\sigma(m) = 2m$; i.e., m is perfect.

Part (ii) also solved by Paul S. Bruckman.

Factorials and Products of Differences of Triangular Numbers

H-662 Proposed by Rigoberto Flórez, Sumter, SC (Vol. 45, No. 4, November 2007)

Let $T_n = n(n+1)/2$ be the nth triangular number.

(i) If $n \geq 1$, show that

$$n! = 2^{\lfloor n/2 \rfloor} \prod_{i=1}^{\lfloor n/2 \rfloor - 1} (T_{\lfloor (n+1)/2 \rfloor} - T_i).$$

(ii) Suppose that T is equal to $\prod_{i=0}^{k-1}(T_k - T_i)$ or $\prod_{i=0}^{k-1}(T_{k+1} - T_i)$. Let p be the first prime number greater than T. Is $p - T$ always equal to one or to a prime number?

Solution of part (i) by the proposer

We first suppose that $n = 2k$. Let $n! = 1 \cdot 2 \cdot 3 \cdots (n-2) \cdot (n-1) \cdot n$. First, we group the first number with the last number in this product, then the second number with the second to the last, and so on. In this way,

$$\begin{aligned}
n! &= (1 \cdot n) \cdot (2 \cdot (n-1)) \cdot (3 \cdot (n-2)) \cdots \left[\frac{n}{2} \cdot \left[n - \left(\frac{n}{2} - 1\right)\right]\right] \\
&= (n) \cdot (2n-2) \cdot (3n-6) \cdots \left(\frac{n}{2} \cdot n - \frac{n}{2}\left(\frac{n}{2} - 1\right)\right) \\
&= \prod_{i=1}^{k} (i(2k) - i(i+1)) = \prod_{i=1}^{k} \left(i(2k) - 2\sum_{j=1}^{i} j\right) = 2^k \prod_{i=1}^{k} \sum_{j=1}^{i} (k-j) \\
&= 2^k \prod_{i=0}^{k-1} \left(\frac{k(k+1)}{2} - \frac{i(i+1)}{2}\right) = 2^k \prod_{i=0}^{k-1} (T_k - T_i).
\end{aligned}$$

Suppose now that $n = 2k+1$. So, $n+1 = 2(k+1)$. By the previous analysis, we have that $(n+1)! = 2^{k+1} \prod_{i=0}^{k}(T_{k+1} - T_i)$. Therefore,

$$\begin{aligned}
n! &= \frac{2^{k+1}}{n+1} \prod_{i=0}^{k}(T_{k+1} - T_i) = \frac{2^{k+1}}{2(k+1)}(T_{k+1} - T_k) \prod_{i=0}^{k-1}(T_{k+1} - T_i) \\
&= \frac{2^{k+1}}{2(k+1)} \left(\frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2}\right) \prod_{i=0}^{k-1}(T_{k+1} - T_i) = 2^k \prod_{i=0}^{k-1}(T_{k+1} - T_i).
\end{aligned}$$

Part (i) also solved by Paul S. Bruckman. No solution to Part (ii) was received.

Fibonacci Numbers and Trigonometric Functions

H-663 Proposed by Charles K. Cook, Sumter, SC (Vol. 45, No. 4, November 2007)

If $n \geq 3$, evaluate $\prod_{j=1}^{F_n - 1} \sin(j\pi/F_n)$.

Solution by the editor

It is well-known (see, for example, formula (24) in [1]), that if $m \geq 2$ is an integer then

$$\prod_{j=1}^{m-1} \sin\left(\frac{\pi j}{m}\right) = \frac{m}{2^{m-1}}.$$

Taking $m = F_n$, we get that the desired product evaluates to $F_n/2^{F_n-1}$.

References

[1] http://mathworld.wolfram.com/TrigonometryAngles.html

Also solved by Paul S. Bruckman, G. C. Greubel, and the proposer.

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