ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-792 Proposed by George A. Hisert, Berkeley, California.

Consider the 3-sequence $T_{i+1} = T_i + T_{i-1} + T_{i-2}$ for all integers i with $T_0 = 0,\ T_1 = T_2 = 1$. Let $S_i = T_i + T_{i-1}$. Prove that for all integers n positive or negative, we have $T_n^2 - T_{n+1}T_{n-1} = T_{-(n+1)}$ and $T_{n+1}T_{n-2} - T_nT_{n-1} = S_{-(n+1)}$.

H-793 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Bogdan Andrei Stanciu, Braşov, Romania.

Let $e_n = (1 + 1/n)^n$. Compute

$$\lim_{n\to\infty}\left(e_{n+1}\sqrt[n+1]{(2n+1)!!F_{n+1}} - e_n\sqrt[n]{(2n-1)!!F_n}\right).$$

Compute the similar limit with all the F's replaced by L's.

H-794 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that

$$\sqrt[3]{\frac{F_n}{5F_{n+2}}} + \sqrt[3]{\frac{F_{n+1}}{5F_{n+2}+3F_{n+1}}} + \sqrt[3]{\frac{F_{n+2}}{5F_{n+2}+3F_n}} < \sqrt[3]{4} \quad \text{for all} \quad n \ge 0.$$

H-795 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{k=1}^{2n}\tan^{-1}\left(\frac{2}{L_{2k-1}}\right) = 2\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{F_{4k-2}}\right).$$

H-796 Proposed by Hideyuki Ohtsuka, Saitama, Japan and Florian Luca, Johannesburg, South Africa.

Find all solutions (x, y) in positive integers of the equation

$$\tan^{-1}\alpha^x - \tan^{-1}\alpha^y = \tan^{-1}x - \tan^{-1}y,$$

where α is the golden section.

SOLUTIONS

Sums of Squares of Members of r-Generalized Fibonacci Like Sequences

H-759 Proposed by H. Ohtsuka, Saitama, Japan. (Vol. 52, No. 3, August 2014)

Let $r \geq 2$ be an integer. Define the sequence $\{G_n\}$ by

$$G_n = G_{n-1} + \cdots + G_{n-r} \qquad (n \geq 1)$$

with arbitrary $G_0, G_1, \ldots, G_{-r+1}$. For an integer $n \geq 1$, prove that

$$\sum_{k=1}^{n} G_k^2 = \sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} \sum_{i=1}^{k} (G_{n+i-k}G_{n+i} - G_{i-k}G_i).$$

Solution by the proposer.

Let a be a root of the characteristic equation

$$x^r - x^{r-1} - x^{r-2} - \cdots - x - 1 = 0. \tag{1}$$

We have

$$a^{r+1} - a^r = (a-1)a^r = (a-1)(a^{r-1} + a^{r-2} + \cdots + a + 1) = a^r - 1.$$

Thus, we have

$$a^{r+1} = 2a^r - 1. \tag{2}$$

Using the identity (2), we have

$$a^{-r} = -a + 2, \quad a^{r+2} = 4a^r - a - 2 \quad \text{and} \quad a^{r+3} = 8a^r - a^2 - 2a - 4. \tag{3}$$

Using WolframAlpha, we have

$$\begin{aligned}
&(a-1)^3 \sum_{k=1}^{r} \Big(k(r-k-1)+2)(a^k - a^{-k}\Big) = (-r+2)a^{r+3} + (3r-4)a^{r+2} \\
&- 2ra^{r+1} - (2ra^2 - 3ra + 4a + r - 2)a^{-r} - 2a^3 + 4a^2 + 4a - 2 \\
&= 2(r-1)(a-1)^3,
\end{aligned}$$

by (2) and (3). That is,

$$\sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} (a^{n+k} - a^{n-k}) = a^n. \tag{4}$$

Note that the characteristic equation (1) has r distinct roots (see [1]). If $a_1, a_2, \ldots, a_r$ are the roots of (1), then we can write

$$G_n = c_1 a_1^n + c_2 a_2^n + \cdots + a_r a_r^n, \tag{5}$$

where the coefficients $c_1, c_2, \dots, c_r$ depend on $G_0,\ G_{-1}, \dots, G_{-r+1}$. By (4) and (5), for $n \geq 1$, we have the identity

$$\sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} (G_{n+k} - G_{n-k}) = G_n. \tag{6}$$

For $n \geq 0$, we have

$$\begin{aligned}
&\sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} \sum_{i=1}^{k} (G_{n+1+i-k}G_{n+1+i} - G_{n+i-k}G_{n+i}) \\
&= \sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} (G_{n+1}G_{n+1+k} - G_{n+1-k}G_{n+1}) \\
&= G_{n+1} \sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} (G_{n+1+k} - G_{n+1-k}) \\
&= G_{n+1}^2,
\end{aligned} \tag{7}$$

by (6). The proof of the desired identity is by mathematical induction on n.

- Letting $n = 0$ in identity (7), we have

$$G_1^2 = \sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} \sum_{i=1}^{k} (G_{1+i-k}G_{1+i} - G_{i-k}G_i).$$

  Thus, the desired identity holds for $n = 1$.
- We assume that the desired identity holds for n. For $n+1$, we have

$$\begin{aligned}
\sum_{k=1}^{n+1} G_k^2 &= G_{n+1}^2 + \sum_{k=1}^{n} G_k^2 \\
&= G_{n+1}^2 + \sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} \sum_{i=1}^{k} (G_{n+i-k}G_{n+i} - G_{i-k}G_i) \\
&= \sum_{k=1}^{r} \frac{k(r-k-1)+2}{2(r-1)} \sum_{i=1}^{k} (G_{n+1+i-k}G_{n+1+i} - G_{i-k}G_i),
\end{aligned}$$

  by (7). Thus, the desired identity holds for $n+1$.

Editor's comment: Kenneth B. Davenport points out that in Theorem 3.1 in [2], Curtis Cooper derived the following formula

$$\sum_{k=0}^{n} G_k^2 + \sum_{i=2}^{r-1} \sum_{k=0}^{n-i} G_k G_{k+i} = G_n G_{n+1},$$

which perhaps can be used to give an alternative proof of the identity of H-759.

References

[1] E. P. Miles, *Generalized Fibonacci numbers and associated matrices*, The American Math. Monthly, **67.8** (1960), 745–752.

[2] C. Cooper, *Two identities involving generalized Fibonacci numbers*, J. Inst. Math. Comput. Sci. Math Ser., **23.1** (2010), 21–26.

Also solved by Dmitry Fleischman.

An Inequality Involving Powers, Binomial Coefficients and Fibonacci Numbers

H-760 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania. (Vol. 52, No. 4, November 2014)

Prove that if $m \geq 1,\ k \geq 1,\ n \geq 0$ are integers then

$$m^m \sum_{p=0}^{2n+1} \left(1 + \sum_{k=0}^{p} \binom{2n+1}{p} \binom{p}{k} F_k\right)^{m+1} \geq 5^n (m+1)^{m+1} L_{2n+1}.$$

Solution by Hideyuki Ohtsuka.

We use the identities

(i) $\displaystyle\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$ (see [1] (47));

(ii) $\displaystyle\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{2i} = 5^n L_{2n+1}$ (see [1](70)).

We have

$$\left(\frac{m}{m+1}\right)^{m+1} \sum_{p=0}^{2n+1} \left(1 + \binom{2n+1}{p} \sum_{k=0}^{p} \binom{p}{k} F_k\right)^{m+1}$$

$$= \sum_{p=0}^{2n+1} \left(1 - \frac{1}{m+1} + \frac{m}{m+1} \binom{2n+1}{p} F_{2p}\right)^{m+1} \quad \text{(by } (i))$$

$$\geq \sum_{p=0}^{2n+1} \left(1 + (m+1)\left(-\frac{1}{m+1} + \frac{m}{m+1} \binom{2n+1}{p} F_{2p}\right)\right) \quad \text{(by Bernoulli's inequality)}$$

$$= m \sum_{p=0}^{2n+1} \binom{2n+1}{p} F_{2p} = 5^n m L_{2n+1} \quad \text{(by } (i)).$$

Therefore, we obtain the desired identity.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, and the proposers.

A Series Whose Sum Involves π, $\ln 2$ and $\zeta(3)$

H-761 Proposed by Ovidiu Furdui, Campia Turzii, Romania. (Vol. 52, No. 4, November 2014)

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right)^2 = \frac{\pi^2 \ln 2}{6} - \frac{\ln^3 2}{3} - \frac{3}{4}\zeta(3).$$

Solution by AN-anduud Problem Solving Group.

We will be using the following four well-known identities:

$$\ln\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx = -\frac{5}{8}\zeta(3),$$

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{1+x}dx = \frac{1}{24}(8\ln^3 2 - 2\pi^2\ln 2 + 3\zeta(3)),$$

$$\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots = \int_0^1 \frac{x^n}{1+x}dx = \ln 2 - n\int_0^1 x^{n-1}\ln(1+x)dx,$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2.$$

It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right)^2 \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^1 \frac{x^n}{1+x}dx\right)\left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right) \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\left(\ln 2 - n\int_0^1 x^{n-1}\ln(1+x)dx\right)\left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right) \\
&= \ln 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right) \\
&\quad - \sum_{n=1}^{\infty}\int_0^1 x^{n-1}\ln(1+x)dx\int_0^1 \frac{y^n}{1+y}dy \\
&= \left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\right)\ln 2 - \int_0^1 \ln(1+x)\left(\int_0^1 \frac{y}{1+y}\sum_{n=1}^{\infty}(xy)^{n-1}dy\right)dx \\
&= \left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\right)\ln 2 - \int_0^1 \ln(1+x)\left(\int_0^1 \frac{y}{(1-xy)(1+y)}dy\right)dx \\
&= \left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\right)\ln 2 \\
&\quad + \int_0^1 \left(\ln(1+x)\left(\frac{\ln(1-x)}{x} - \frac{\ln(1-x)}{1+x} + \frac{\ln 2}{1+x}\right)\right)dx \\
&= \left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\right)\ln 2 + \int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx \\
&\quad - \int_0^1 \frac{\ln(1-x)\ln(1+x)}{1+x} + \ln 2\int_0^1 \frac{\ln(1+x)}{1+x}dx \\
&= \left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\right)\ln 2 - \frac{5}{8}\zeta(3) - \frac{1}{24}(8\ln^3 2 - 2\pi^2\ln 2 + 3\zeta(3)) + \frac{1}{2}\ln^3 2.
\end{aligned}$$

The last expression simplifies to the desired answer

$$\frac{\pi^2}{6}\ln 2 - \frac{1}{3}\ln^3 2 - \frac{3}{4}\zeta(3).$$

Also solved by Khristo N. Boyadzhiev, G. C. Greubel, Anastasios Kotronis, Albert Stadler, and the proposer.

Identities with Sums of Powers of Fibonacci Numbers and Binomial Coefficients

H-762 Proposed by George Hisert, Berkeley, California. (Vol. 52, No. 4, November 2014)

Prove that for any positive integers r and n and positive integer p,

$$\text{(i)}\quad \sum_{k=0}^{\lfloor (p-1)/2\rfloor} (-1)^k \binom{p}{k} F_{2(p-2k)r}(F_{n+4r}^{p-k}F_n^k - (-1)^p F_{n+4r}^k F_n^{p-k}) = F_{4r}^p F_{p(n+2r)};$$

$$\text{(ii)}\quad \sum_{k=0}^{\lfloor (p-1)/2\rfloor} (-1)^k \binom{p}{k} F_{2(p-2k)r}(L_{n+4r}^{p-k}L_n^k - (-1)^p L_{n+4r}^k L_n^{p-k}) = F_{4r}^p L_{p(n+2r)}.$$

Solution by Hideyuki Ohtsuka.

Identity (ii) is not correct. We will prove identity (i). We have

$$\begin{aligned}(\alpha^{2r}F_{n+4r} - \alpha^{-2r}F_n)^p &= \sum_{k=0}^{p}\binom{p}{k}(\alpha^{2r}F_{n+4r})^{p-k}(-\alpha^{-2r}F_n)^k \\ &= \sum_{k=0}^{p}(-1)^k\binom{p}{k}\alpha^{2r(p-2k)}F_{n+4r}^{p-k}F_n^k,\end{aligned}$$

and

$$\begin{aligned}(\alpha^{2r}F_{n+4r} - \alpha^{-2r}F_n)^p &= \left(\frac{\alpha^{2r}(\alpha^{n+4r}-\beta^{n+4r}) - \alpha^{-2r}(\alpha^n-\beta^n)}{\sqrt{5}}\right)^p \\ &= \left(\frac{(\alpha^{n+6r}-\beta^{n+2r}) - (\alpha^{n-2r}-\beta^{n+2r})}{\sqrt{5}}\right)^p \\ &= \left(\frac{\alpha^{n+2r}(\alpha^{4r}-\alpha^{-4r}}{\sqrt{5}}\right)^p \\ &= \alpha^{p(n+2r)}F_{4r}^p.\end{aligned}$$

Thus,

$$\sum_{k=0}^{p}(-1)^k\binom{p}{k}\alpha^{2r(p-2k)}F_{n+4r}^{p-k}F_n^k = \alpha^{p(n+2r)}F_{4r}^p.$$

In the same manner,

$$\sum_{k=0}^{p}(-1)^k\binom{p}{k}\beta^{2r(p-2k)}F_{n+4r}^{p-k}F_n^k = \beta^{p(n+2r)}F_{4r}^p.$$

Therefore,

$$\sum_{k=0}^{p}(-1)^k\binom{p}{k}F_{2r(p-2k)}F_{n+4r}^{p-k}F_n^k = F_{p(n+2r)}F_{4r}^p. \tag{1}$$

We have

$$\begin{aligned}
&\sum_{k=\lfloor (p+1)/2\rfloor}^{p} (-1)^k \binom{p}{k} F_{2r(p-2k)} F_{n+4r}^{p-k} F_n^k \\
&= \sum_{k=0}^{\lfloor (p-1)/2\rfloor} (-1)^{p-k} \binom{p}{p-k} F_{2r(p-2(p-k))} F_{n+4r}^{k} F_n^{p-k} \\
&= -\sum_{k=0}^{\lfloor (p-1)/2\rfloor} (-1)^{p-k} \binom{p}{k} F_{2r(p-2k)} F_{n+4r}^{k} F_n^{p-k}, \qquad (2)
\end{aligned}$$

since $F_{2r(p-2(p-k))} = F_{-2r(p-2k)} = -F_{2r(p-2k)}$. The left-hand side of (1) is

$$\begin{aligned}
&\sum_{k=0}^{\lfloor (p-1)/2\rfloor} (-1)^k \binom{p}{k} F_{2r(p-2k)} F_{n+4r}^{p-k} F_n^k + \sum_{k=\lfloor (p+1)/2\rfloor}^{p} (-1)^k \binom{p}{k} F_{2r(p-2k)} F_{n+4r}^{p-k} F_n^k \\
&= \sum_{k=0}^{\lfloor (p-1)/2\rfloor} (-1)^k \binom{p}{k} F_{2r(p-2k)} (F_{n+4r}^{p-k} F_n^k - (-1)^p F_{n+4r}^{k} F_n^{p-k}),
\end{aligned}$$

by (2). Therefore, we obtain (i).

Editor's comment: The proposer noted that the case $p = 7$ of (i) is Advanced Problem H-324, which inspired him to propose the present generalization.

Also solved by the proposer.

Errata: The right hand–side of the identity proposed at H-762 (ii) should be $5^{p/2} F_{4r}^p F_{p(n+2r)}$ for even p and $5^{(p-1)/2} F_{4r}^p L_{p(n+2r)}$ for odd p. The editor and proposer thank Hideyuki Ohtsuka for this correction.

Late Acknowledgement. Adnan A. Ali solved H-752 and Kenneth B. Davenport solved H-758.