ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by email at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-878 Proposed by Robert Frontczak, Stuttgart, Germany

Prove that for all $n \geq 1$:

$$\sum_{k=1}^{n} L_k^3 L_{k+1}^3 = \frac{1}{9}\left(\left(\frac{5}{2}L_{3(n+1)} - L_{n+1}^3\right)^2 - 81\right).$$

H-879 Proposed by Robert Frontczak, Stuttgart, Germany

Prove the following identities for the Fibonacci and Lucas numbers

$$\begin{aligned}\sqrt{5}(F_{2n} - F_n) &= \sum_{k=1}^{n} \binom{n}{k}(k+1)^{k-1}((\alpha - k)^{n-k} - (\beta - k)^{n-k}) \\ &= \sum_{k=1}^{n} \binom{n}{k}(-1)^{k-1}(k-1)^{k-1}((\alpha + k)^{n-k} - (\beta + k)^{n-k})\end{aligned}$$

and

$$\begin{aligned}L_{2n} - L_n &= \sum_{k=1}^{n} \binom{n}{k}(k+1)^{k-1}((\alpha - k)^{n-k} + (\beta - k)^{n-k}) \\ &= \sum_{k=1}^{n} \binom{n}{k}(-1)^{k-1}(k-1)^{k-1}((\alpha + k)^{n-k} + (\beta + k)^{n-k}).\end{aligned}$$

H-880 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain

For any positive integer k, the Fibonacci k-sequence $\{F_{k,n}\}_{n\geq 0}$ is defined by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$ with $F_{k,0} = 0$, $F_{k,1} = 1$. Prove that

$$\sum_{i=0}^{n} \binom{2n+1}{n-i} F_{k,2i+1} = (k^2+4)^n.$$

H-881 **Proposed by Hideyuki Ohtsuka, Saitama, Japan**

For any positive integers r and n, prove that

$$\sum_{k=0}^{n}\binom{2n}{n-k}\frac{F_{4rk}}{F_{4r}}=\sum_{k=0}^{n-1}\binom{2k}{k}L_{2r}^{2n-2k-2}.$$

H-882 **Proposed by Robert Frontczak, Stuttgart, Germany**

Prove the following identities for the Fibonacci and Lucas numbers

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{k+1}=\frac{F_{2n+1}+L_{2n+1}}{n+1}\quad\text{and}\quad\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{(k+1)(k+2)}=\frac{F_{2n+2}+L_{2n+2}-2}{(n+1)(n+2)}.$$

Errata. In the right–hand side of **H-868**, the term to be summed inside the most inner sum should be

$$\frac{k^{2\ell}(2^{2\ell}-1)}{\ell d^{2\ell}}\zeta(2\ell)\quad\text{instead of}\quad\frac{k^{2\ell}(2^{2\ell}-1)}{\ell 2^{2\ell}}\zeta(2\ell).$$

SOLUTIONS

Identities with generalized-balancing numbers

H-844 **Proposed by Robert Frontczak, Stuttgart, Germany**
(Vol. 57, No. 3, August 2019)

Let $B_n=B_n(\alpha,\beta)$ be a generalized balancing number given by $B_0(\alpha,\beta)=\alpha,\ B_1(\alpha,\beta)=\beta$, and for $n\geq 2$,

$$B_n(\alpha,\beta)=6B_{n-1}(\alpha,\beta)-B_{n-2}(\alpha,\beta).$$

Prove that

$$\sum_{k=0}^{2n}\binom{4n}{2k}B_{2k}(\alpha,\beta)=(2^{6n-1}+2^{4n-1})B_{2n}(\alpha,\beta)$$

and

$$\sum_{k=0}^{\lfloor(4n-1)/2\rfloor}\binom{4n}{2k+1}B_{2k}(\alpha,\beta)=(2^{6n-1}-2^{4n-1})B_{2n}(\alpha,\beta).$$

Solution by Ángel Plaza, Gran Canaria, Spain

We will use the following identities for the generalized balancing numbers

$$\sum_{k=0}^{m}\binom{2m}{2k}B_{2k}(\alpha,\beta)=(2^{3m-1}+2^{2m-1})B_m(\alpha,\beta)$$

and

$$\sum_{k=0}^{\lfloor(2m-1)/2\rfloor}\binom{2m}{2k+1}B_{2k+1}(\alpha,\beta)=(2^{3m-1}-2^{2m-1})B_m(\alpha,\beta),$$

which are respectively identities (2.16) and (2.17) in Proposition 2.1 in [1]. By letting $m=2n$, the proposed identities follow.

Reference

[1] R. Frontczak, *Identities for generalized balancing numbers*, Notes on Number Theory and Discrete Mathematics, **25** (2019), 169–180.

Also solved by Brian Bradie, David Terr, and the proposer.

Some double limits with ratios of Lucas numbers

H-845 Proposed by D. M. Bătineţu, Bucharest, Romania, and N. Stanciu, Buzău, Romania (Vol. 57, No. 3, August 2019)

Compute

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{L_n/((x+1)l_{n+1})}-(f(x))^{L_n/(xL_{n+1})}\right)x^{L_{n-1}/L_{n+1}}\right),$$

where $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a function that satisfies $\lim_{x\to\infty} f(x+1)/(xf(x))=a\in\mathbb{R}_+^*$.

Solution by the proposers

We denote $u_n=L_n/L_{n+1}$. We have

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n+1}+\beta^{n+1}}=\frac{1}{\alpha}.$$

We also have

$$\begin{aligned}\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x} &= \lim_{n\to\infty}\frac{f(n)^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{f(n)}{n^n}}\\ &= \lim_{n\to\infty}\frac{f(n+1)}{(n+1)^{n+1}}\cdot\frac{n^n}{f(n)}=\lim_{n\to\infty}\frac{f(n+1)}{nf(n)}\left(\frac{n}{n+1}\right)^{n+1}=\frac{a}{e}.\end{aligned}$$

We denote

$$v(x)=\frac{f(x+1)^{\frac{L_n}{(x+1)L_{n+1}}}}{f(x)^{\frac{L_n}{xL_{n+1}}}}=\left(\frac{f(x+1)^{\frac{1}{x+1}}}{f(x)^{\frac{1}{x}}}\right)^{u_n}.$$

We have $\lim_{x\to\infty}v(x)=1$, so

$$\lim_{x\to\infty}\frac{v(x)-1}{\ln v(x)}=1$$

and

$$\lim_{x\to\infty}(v(x))^x=\lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\cdot\frac{1}{f(x+1)^{\frac{1}{x+1}}}\right)^{u_n}=\lim_{x\to\infty}\left(\frac{a(x+1)}{(f(x+1))^{\frac{1}{x+1}}}\right)^{u_n}=e^{u_n},$$

therefore

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}(v(x))^x\right)=e^{\frac{1}{\alpha}}.$$

Hence,

$$
\begin{aligned}
& \lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{\frac{L_n}{(x+1)L_{n+1}}}-(f(x))^{\frac{L_n}{xL_{n+1}}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)\\
= & \lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{\frac{u_n}{x+1}}-(f(x))^{\frac{u_n}{x}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)\\
= & \lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{\frac{u_n}{x+1}}-(f(x))^{\frac{u_n}{x}}\right)x^{\frac{L_{n+1}-L_n}{L_{n+1}}}\right)\\
= & \lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x)^{\frac{u_n}{x}}\right)\left((v(x)-1)x^{1-u_n}\right)\right)\\
= & \lim_{n\to\infty}\left(\lim_{x\to\infty}\left(f(x)^{\frac{u_n}{x}}\right)\left(\frac{v(x)-1}{\ln v(x)}\right)x^{1-u_n}\ln v(x)\right)\\
= & \lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\left(\frac{f(x)^{\frac{1}{x}}}{x}\right)^{\frac{1}{u_n}}\left(\frac{v(x)-1}{\ln v(x)}\right)\ln(v(x)^x)\right)\right)\\
= & \lim_{n\to\infty}\left(\left(\frac{a}{e}\right)^{u_n}\cdot 1\cdot\ln(e)^{u_n}\right)=\left(\frac{a}{e}\right)^{\frac{1}{\alpha}}\ln(e^{\frac{1}{\alpha}})\\
= & \left(\frac{a}{e}\right)^{\frac{1}{\alpha}}\left(\frac{1}{\alpha}\right).
\end{aligned}
$$

Editorial comment. The original problem had exponent "L_n/xF_{n+1}" on $f(x+1)$. With this exponent, the given limit is infinite. Due to an editorial mishap for which the editor apologizes, this problem was republished as **H-854** again with the wrong exponent, which was corrected in an Errata in issue **59.1**. **Brian Bradie** sent in a more general solution to **H-845** with the correct exponent in which he found the formula for the above double limit when the sequence $\{L_n\}_{n\geq 0}$ is replaced by the sequence $\{G_{k,n}\}_{n\geq 0}$ of initial terms $G_{k,0}=g_0$, $G_{k,1}=g_1$, and recurrence $G_{k,n+1}=kG_{k,n}+G_{k,n-1}$ for $n\geq 1$. Assuming in this case that $f(x)$ satisfies $\lim_{x\to\infty} f(x+1)/(x^k f(x))=a\in\mathbb{R}_+^*$, he shows that the resulting limit is

$$\left(\frac{a}{e^k}\right)^{\frac{1}{\alpha_k}}\left(\frac{k}{\alpha_k}\right),$$

where $\alpha_k=(k+\sqrt{k^2+4})/2$. Problem **H-854** was also solved by **Albert Stadler**.

A formula for the convolution of Tribonacci and triangular numbers

H-846 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 57, No. 4, November 2019)

Let $T_n=n(n+1)/2$ be the nth triangular number. Let $(G_n)_{n\geq 0}$ be the Tribonacci sequence given by $G_0=0$, $G_1=G_2=1$, and for $n\geq 3$,

$$G_n=G_{n-1}+G_{n-2}+G_{n-3}.$$

Prove that

$$\sum_{k=0}^{n}T_kG_{n-k}=\frac{1}{2}(G_{n+4}-T_{n+4})+(n+3).$$

Extend this result to a generalized Tribonacci sequence with $G_0=a$, $G_1=b$, and $G_2=c$.

Solution by Jason L. Smith, Richland Community College

The method of generating functions will be helpful. The generating functions for the triangular numbers [3], Tribonacci numbers [2] and generalized Tribonacci numbers [1], respectively are

$$\begin{aligned} T(x) &= \sum_{k=0}^{\infty} T_k x^k = \frac{x}{(1-x)^3}; \\ G(x) &= \sum_{k=0}^{\infty} G_k x^k = \frac{x}{1-x-x^2-x^3}; \\ H(x) &= \sum_{k=0}^{\infty} H_k x^k = \frac{a+(b-a)x+(c-b-a)x^2}{1-x-x^2-x^3}. \end{aligned}$$

In the most general case, the sum we seek is the coefficient of x^n in the expansion of the product $T(x)H(x)$,

$$T(x)H(x) = \frac{x}{(1-x)^3} \cdot \frac{a+(b-a)x+(c-b-a)x^2}{1-x-x^2-x^3}.$$

This expression can be expanded in partial fractions as

$$\begin{aligned} T(x)H(x) &= \frac{-(\frac{1}{2}a+b+c)+(\frac{3}{2}a+\frac{3}{2}b+c)x-\frac{1}{2}(a+b+c)x^2}{(1-x)^3} \\ &+ \frac{(\frac{1}{2}a+b+c)+(\frac{1}{2}a+\frac{1}{2}b+c)x+\frac{1}{2}(a+b+c)x^2}{1-x-x^2-x^3}. \end{aligned}$$

Using the formulas for $T(x)$ and $G(x)$ above, we get that

$$\begin{aligned} T(x)H(x) &= T(x)\left[\frac{-\frac{1}{2}(a+b+c)}{x} + \left(\frac{3}{2}a+\frac{3}{2}b+c\right) - \frac{1}{2}(a+b+c)x\right] \\ &+ G(x)\left[\frac{(\frac{1}{2}a+b+c)}{x} + \left(\frac{1}{2}a+\frac{1}{2}b+c\right) + \frac{1}{2}\left(a+b+c\right)x\right]. \end{aligned}$$

From this, we can glean the coefficients of x^n on each side:

$$\begin{aligned} \sum_{k=0}^{n} T_k H_{n-k} &= -\left(\frac{1}{2}a+b+c\right)T_{n+1} + \left(\frac{3}{2}a+\frac{3}{2}b+c\right)T_n - \frac{1}{2}\left(a+b+c\right)T_{n-1} \\ &+ \left(\frac{1}{2}a+b+c\right)G_{n+1} + \left(\frac{1}{2}a+\frac{1}{2}b+c\right)G_n + \frac{1}{2}\left(a+b+c\right)G_{n-1} \\ &= \frac{1}{2}a\left(G_{n+1}+G_n+G_{n-1}-T_{n+1}+3T_n-T_{n-1}\right) \\ &+ \frac{1}{2}b\left(2G_{n+1}+G_n+G_{n-1}-2T_{n+1}+3T_n-T_{n-1}\right) \\ &+ \frac{1}{2}c\left(2G_{n+1}+2G_n+G_{n-1}-2T_{n+1}+2T_n-T_{n-1}\right). \end{aligned}$$

The standard recursions for the Tribonacci numbers ($G_n = G_{n-1}+G_{n-2}+G_{n-3}$) and the triangular numbers ($T_n = T_{n-1}+n$) can be used in various forms or repeatedly to obtain the formula

$$\begin{aligned}\sum_{k=0}^{n} T_k H_{n-k} &= \frac{1}{2}a\left(G_{n+2}+T_{n+4}-4n-11\right)\\ &+ \frac{1}{2}b\left(G_{n+4}-G_{n+3}-n-2\right)\\ &+ \frac{1}{2}c\left(G_{n+3}-T_{n+4}+3n+8\right).\end{aligned}$$

When $a=0$, $b=c=1$, we get

$$\sum_{k=0}^{n} T_k H_{n-k} = \frac{1}{2}(G_{n+4}-T_{n+4})+(n+3).$$

References

[1] R. Frontczak, *Relations for generalised Fibonacci and Tribonacci sequences*, Notes on Number Theory and Discrete Math, **25** (2019), 178–192. [2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons Inc., New York, NY, 2001, p. 532.

[3] E. W. Weisstein, *Triangular number*, `http://mathworld.wolfram.com/TriangularNumber.html`.

Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Raphael Schumacher, Albert Stadler, and the proposer.

A closed form for the convolution of Balancing numbers and Fibonacci cubes

H-847 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 57, No. 4, November 2019)

Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers. Let $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ be the balancing and Lucas-balancing numbers, respectively; i.e., for all $n \geq 1$ we have

$$B_{n+1} = 6B_n - B_{n-1} \qquad \text{and} \qquad C_{n+1} = 6C_n - C_{n-1},$$

with $B_0 = 0$, $B_1 = 1$, $C_0 = 1$, $C_1 = 3$. Show that for all $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^{k+1} B_{n-k} F_{2k}^3 = \frac{(-1)^n}{24} F_{2(n-1)} F_{2n} F_{2(n+1)},$$

and

$$\sum_{k=0}^{n} (-1)^{k} C_{n-k} F_{2k}^3 = \frac{(-1)^n}{8} \left(F_{2(n-1)} F_{2n} F_{2(n+1)} + \frac{F_{2n} F_{2(n+1)} F_{2(n+2)}}{3} \right).$$

Solution by Raphael Schumacher, ETH Zurich, Switzerland

The generating function of the balancing numbers is given by

$$f_1(x) := \sum_{k=0}^{\infty} B_k x^k = \frac{x}{x^2-6x+1}$$

and the generating function of the Lucas-balancing numbers is

$$f_2(x) := \sum_{k=0}^{\infty} C_k x^k = \frac{1-3x}{x^2-6x+1}.$$

We have also that

$$f_3(x) := \sum_{k=0}^{\infty} F_{2k}^3 x^k = \frac{x(x^2+6x+1)}{(x^2-3x+1)(x^2-18x+1)},$$

$$f_4(x) := \sum_{k=0}^{\infty} F_{2(k-1)}F_{2k}F_{2(k+1)} x^k = \frac{24x^2}{(x^2-3x+1)(x^2-18x+1)},$$

$$f_5(x) := \sum_{k=0}^{\infty} F_{2k}F_{2(k+1)}F_{2(k+2)} x^k = \frac{24x}{(x^2-3x+1)(x^2-18x+1)},$$

and we calculate that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{k+1}B_{n-k}F_{2k}^3\right)x^n = -f_1(x)f_3(-x) = \frac{1}{24}f_4(-x)$$
$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{24}F_{2(n-1)}F_{2n}F_{2(n+1)}\right)x^n$$

and that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{k}C_{n-k}F_{2k}^3\right)x^n = f_2(x)f_3(-x) = \frac{1}{8}f_4(-x) + \frac{1}{24}f_5(-x)$$
$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{8}\left(F_{2(n-1)}F_{2n}F_{2(n+1)} + \frac{F_{2n}F_{2(n+1)}F_{2(n+2)}}{3}\right)\right)x^n.$$

The above two calculations prove the two desired formulas.

Also solved by Brian Bradie, Dmitry Fleischman, Ángel Plaza, Albert Stadler, and the proposer.

A nested radical

H-848 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 57, No. 4, November 2019)

Let $p \geq 2$ be a real number. The sequence $(U_n)_{n\geq 0}$ is defined by

$$U_0 = 0, \quad U_1 = 1, \quad \text{and} \quad U_{n+2} = pU_{n+1} - U_n \quad \text{for all} \quad n \geq 0.$$

Prove that

$$\lim_{n\to\infty}\sqrt{1+U_1\sqrt{1+U_2\sqrt{1+U_3\sqrt{\cdots+U_{n-1}\sqrt{1+U_n}}}}} = p.$$

Solution by Raphael Schumacher, ETH Zurich, Switzerland

We have that $U_2 = p$ and that

$$U_n^2 - 1 = U_{n+1}U_{n-1} \text{ for all } n \in \mathbb{N},$$

because we have that

$$U_1^2 - 1 = 1 - 1 = 0 = p \cdot 0 = U_2U_0,$$

and we can calculate by induction that

$$\begin{aligned}
U_{n+1}^2 - 1 &= (pU_n - U_{n-1})^2 - 1 \\
&= p^2U_n^2 - 2pU_nU_{n-1} + U_{n-1}^2 - 1 \\
&= p^2U_n^2 - pU_{n-1}U_n - pU_nU_{n-1} + U_{n-1}^2 - 1 \\
&= p^2U_n^2 - pU_{n-1}U_n - (pU_n - U_{n-1})\,U_{n-1} - 1 \\
&= p^2U_n^2 - pU_{n-1}U_n - U_{n+1}U_{n-1} - 1 \\
&= p^2U_n^2 - pU_{n-1}U_n - (U_{n+1}U_{n-1} + 1) \\
&= p^2U_n^2 - pU_{n-1}U_n - U_n^2 \\
&= p\,(pU_n - U_{n-1})\,U_n - U_n^2 \\
&= pU_{n+1}U_n - U_n^2 \\
&= (pU_{n+1} - U_n)\,U_n \\
&= U_{n+2}U_n \;\text{ for all } n \in \mathbb{N}.
\end{aligned}$$

Therefore, we get that

$$\begin{aligned}
p &= \sqrt{U_2^2} \\
&= \sqrt{1 + U_1U_3} = \sqrt{1 + U_1\sqrt{U_3^2}} \\
&= \sqrt{1 + U_1\sqrt{1 + U_2U_4}} = \sqrt{1 + U_1\sqrt{1 + U_2\sqrt{U_4^2}}} \\
&= \sqrt{1 + U_1\sqrt{1 + U_2\sqrt{1 + U_3U_5}}} = \dots \\
&= \sqrt{1 + U_1\sqrt{1 + U_2\sqrt{1 + U_3\sqrt{\dots + U_{n-1}\sqrt{1 + U_nU_{n+2}}}}}}.
\end{aligned}$$

Therefore, we conclude that

$$\sqrt{1 + U_1\sqrt{1 + U_2\sqrt{1 + U_3\sqrt{\dots + U_{n-1}\sqrt{1 + U_n}}}}} < p \quad \text{for all } n \in \mathbb{N}.$$

From the formula

$$\sum_{n=0}^{\infty} U_nx^n = \frac{x}{x^2 - px + 1},$$

we obtain the explicit formulas

$$U_n = \frac{\left(p + \sqrt{p^2 - 4}\right)^n - \left(p - \sqrt{p^2 - 4}\right)^n}{2^n\sqrt{p^2 - 4}} \le p^{n-1}, \;\text{ if } p > 2 \text{ (from the recurrence relation)},$$

$$U_n = n, \;\text{ if } p = 2.$$

We will now follow [1] and [2]. To see that the limit is equal to p, we let $\varepsilon \in (0,p)$ and set $s = 1 - \frac{\varepsilon}{p}$. We then have

$$\begin{aligned} p - \varepsilon = sp &= s\sqrt{1 + U_1\sqrt{1 + U_2\sqrt{1 + U_3\sqrt{\ldots + U_{n-1}\sqrt{1 + U_nU_{n+2}}}}}} \\ &= \sqrt{s^2 + U_1\sqrt{s^4 + U_2\sqrt{s^8 + U_3\sqrt{\ldots + U_{n-1}\sqrt{s^{2^n}\left(1 + U_nU_{n+2}\right)}}}}} \\ &< \sqrt{1 + U_1\sqrt{1 + U_2\sqrt{1 + U_3\sqrt{\ldots + U_{n-1}\sqrt{s^{2^n}\left(1 + U_nU_{n+2}\right)}}}}}, \end{aligned}$$

where for the last inequality, we have used that $s \in (0,1)$.

Let $n \geq 6$. Because $s \in (0,1)$, there exists N_ε such that for all $n > N_\varepsilon + 6$ we have

$$s^{2^n} < s^{n(n+2)} \leq \frac{1}{p^{n+2}} \leq \frac{1}{U_{n+3}} \leq \frac{U_n}{1 + U_nU_{n+2}} = \frac{1}{\frac{1}{U_n} + U_{n+2}}$$

and for example $N_\varepsilon = -\log(p)/\log(s)$ if $p > 2$ and

$$s^{2^n} < s^{n^2} \leq \frac{1}{2^n} \leq \frac{1}{n+3} \leq \frac{U_n}{1 + U_nU_{n+2}} = \frac{n}{1 + n(n+2)} = \frac{1}{\frac{1}{n} + n + 2}$$

and $N_\varepsilon = -\log(2)/\log(s)$ if $p = 2$, because we have that

$$\begin{aligned} &2^n > n(n+2) \ \text{ for all } n \geq 6, \\ &2^n > n^2 \ \text{ for all } n \geq 5, \\ &2^n > n + 3 \ \text{ for all } n \geq 3. \end{aligned}$$

Thus, for $n > N_\varepsilon$, we have that

$$\begin{aligned} \sqrt{1 + U_1\sqrt{1 + U_2\sqrt{\ldots + U_{n-1}\sqrt{1 + U_n}}}} &> \sqrt{1 + U_1\sqrt{1 + U_2\sqrt{\ldots + U_{n-1}\sqrt{s^{2^n}\left(1 + U_nU_{n+2}\right)}}}} \\ &> p - \varepsilon. \end{aligned}$$

Therefore, we conclude finally that

$$p = \lim_{n\to\infty}\left(\sqrt{1 + U_1\sqrt{1 + U_2\sqrt{1 + U_3\sqrt{\ldots + U_{n-1}\sqrt{1 + U_n}}}}}\right),$$

which is the claimed formula.

References

[1] https://www.fq.math.ca/Problems/February2017advanced.pdf.
[2] https://www.fq.math.ca/Problems/AdvProbMay2018.pdf.

Also solved by Dmitry Fleischman, Albert Stadler, and the proposer.