ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2019. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1231 Proposed by Kenny B. Davenport, Dallas, PA.

Find the closed form expressions for the sums

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{nF_n}{8^n}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \binom{2n}{n} \frac{nL_n}{8^n}.$$

B-1232 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Let $e_n = \left(1 + \frac{1}{n}\right)^n$. Prove that

$$\left(\sum_{i=1}^{n} e_i F_i^2\right)\left(\sum_{j=1}^{n} \frac{F_j^2}{e_j}\right) \le \frac{(e+2)^2}{8e}\, F_n^2 F_{n+1}^2,$$

for any positive integer n.

B-1233 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For any positive integer n, prove that

$$\sum_{k=1}^{n} F_{2k^2}F_{2k} = F_{n(n+1)}^2, \qquad \text{and} \qquad \sum_{k=1}^{n} F_{2F_k^2}F_{2F_{2k}} = F_{2F_nF_{n+1}}^2.$$

B-1234 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

Let $n \geq 3$ be an odd integer. Find the real solutions of the following system of equations:

$$\begin{aligned} x_1^3 + x_1 + x_2 &= F_1x_1^2 + F_3, \\ x_2^5 + x_2 + x_3 &= F_2x_2^4 + F_4, \\ &\vdots \\ x_{n-1}^{2n-1} + x_{n-1} + x_n &= F_{n-1}x_{n-1}^{2n-2} + F_{n+1}, \\ x_n^{2n+1} + \frac{F_{n+2}-1}{F_n}x_n + x_1 &= F_nx_n^{2n} + F_{n+2}. \end{aligned}$$

B-1235 **Proposed by Kenny B. Davenport, Dallas, PA.**

Prove that, for any integer $n \geq 1$,

$$\sum_{k=1}^{n} F_{k-1}F_kF_{k+1} = \frac{1}{3}\left(F_{n-1}^3 + F_n^3 + F_{n+1}^3 - \frac{F_{3n-1}+3}{2}\right).$$

SOLUTIONS

A Lucas Inequality

B-1194 **(Corrected) Proposed by D. M. Bătineţu-Giurgui, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**
(Vol. 55.3, August 2017)

Prove that

$$\frac{L_1}{(L_1^2+L_2^2+2)^{m+1}} + \frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^{m+1}} + \cdots + \frac{L_n}{(L_1^2+L_2^2+\cdots+L_{n+1}^2+2)^{m+1}} \geq \frac{L_{n+2}-3}{(3L_{n+2})^{m+1}}.$$

for any positive integers n and m.

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC.

Since $\sum_{i=1}^n L_i^2 = L_nL_{n+1} - 2$, the proposed inequality can be written as

$$\frac{L_1}{(L_2L_3)^{m+1}} + \frac{L_2}{(L_3L_4)^{m+1}} + \cdots + \frac{L_n}{(L_{n+1}L_{n+2})^{m+1}} \geq \frac{L_{n+2}-3}{(3L_{n+2})^{m+1}}.$$

We use induction to prove the above inequality. When $n = 1$, $\frac{L_1}{(L_2L_3)^{m+1}} = \frac{1}{12^{m+1}} = \frac{L_3-3}{(3L_3)^{m+1}}$, the above claim is true. Assume that the inequality is true when $n = k$ for some arbitrary but fixed integer $k \geq 1$. Then,

$$\frac{L_1}{(L_2L_3)^{m+1}} + \frac{L_2}{(L_3L_4)^{m+1}} + \cdots + \frac{L_{k+1}}{(L_{k+2}L_{k+3})^{m+1}} \geq \frac{L_{k+2}-3}{(3L_{k+2})^{m+1}} + \frac{L_{k+1}}{(L_{k+2}L_{k+3})^{m+1}}.$$

It suffices to prove that

$$\frac{L_{k+2}-3}{(3L_{k+2})^{m+1}} + \frac{L_{k+1}}{(L_{k+2}L_{k+3})^{m+1}} \geq \frac{L_{k+3}-3}{(3L_{k+3})^{m+1}},$$

or equivalently,

$$L_{k+3}^{m+1}(L_{k+2}-3) + 3^{m+1}L_{k+1} \geq L_{k+2}^{m+1}(L_{k+3}-3).$$

The last inequality can be rearranged further to

$$(L_{k+3}^{m+1} - L_{k+2}^{m+1})(L_{k+2}-3) \geq L_{k+1}(L_{k+2}^{m+1} - 3^{m+1}) = (L_{k+3} - L_{k+2})(L_{k+2}^{m+1} - 3^{m+1}),$$

or

$$L_{k+3}^m + L_{k+3}^{m-1}L_{k+2} + \cdots + L_{k+2}L_{k+2}^{m-1} + L_{k+2}^m \geq L_{k+2}^m + L_{k+1}^{m-1} \cdot 3 + \cdots + L_{k+2} \cdot 3^{m-1} + 3^m,$$

which is clearly true when $k \geq 1$. Therefore, according to the principle of mathematical induction, the claimed inequality is true for all positive integers n.

Editor's Notes: This problem is similar to Problem B-1173(ii).

Also solved by Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, and the proposer.

A Problem with Many Solutions

B-1211 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 55.3, August 2017)

For $n \geq 1$, prove that

$$F_{n-1}^3 + \sum_{k=1}^{n} F_k^3 = \frac{F_{3n-1}+1}{2}.$$

Solution 1 by Donghae Lee (student), Masan Jeil Girls' Middle School and Institute of Gifted Education in Science, Kyungnam University, Changwon, South Korea.

We use induction. For $n = 1$, the equality clearly holds. Assume it holds for some integer $n \geq 1$, we want to show that

$$F_n^3 + \sum_{k=1}^{n+1} F_k^3 = \frac{F_{3n+2}+1}{2}.$$

Thus, it is enough to prove that

$$\left(F_n^3 + \sum_{k=1}^{n+1} F_k^3\right) - \left(F_{n-1}^3 + \sum_{k=1}^{n} F_k^3\right) = \frac{F_{3n+2}+1}{2} - \frac{F_{3n-1}+1}{2},$$

which is equivalent to

$$F_n^3 + F_{n+1}^3 - F_{n-1}^3 = \frac{F_{3n+2} - F_{3n-1}}{2} = \frac{F_{3n+1} + F_{3n} - F_{3n-1}}{2} = F_{3n}.$$

Using the identity $F_k = F_i F_{k+1-i} + F_{i-1} F_{k-i}$, we obtain

$$\begin{aligned} F_{3n} &= F_{n+1}F_{2n} + F_n F_{2n-1} \\ &= F_{n+1}(F_n F_{n+1} + F_{n-1}F_n) + F_n(F_n^2 + F_{n-1}^2) \\ &= F_n(F_{n+1}^2 + F_{n+1}F_{n-1} + F_{n-1}^2) + F_n^3 \\ &= (F_{n+1} - F_{n-1})(F_{n+1}^2 + F_{n+1}F_{n-1} + F_{n-1}^2) + F_n^3 \\ &= F_{n+1}^3 - F_{n-1}^3 + F_n^3, \end{aligned}$$

and the proof is completed.

Solution 2 by David Galante (student) and Ángel Plaza (jointly), Universidad de Las Palmas de Gran Canaria, Spain.

The proposed identity may be written for $n \geq 0$ as

$$F_n^3 + \sum_{k=1}^{n+1} F_k^3 = \frac{F_{3n+2} + 1}{2}.$$

We will show that the sequences on both sides of the equation have the same generating function.

Let $F(x)$ the generating function corresponding to the left side of the equation, and $g(x)$ the generating function of $\left\{F_n^3\right\}_{n\geq 0}$. It is well-known [1, equation (4.12)] that $g(x) = \frac{x(1-2x-x^2)}{(1+x-x^2)(1-4x-x^2)}$. Thus, the generating function of $\left\{\sum_{k=0}^{n} F_k^3\right\}_{n\geq 0}$ is $h(x) = \frac{g(x)}{1-x}$, and consequently the generating function of $\left\{\sum_{k=1}^{n+1} F_k^3\right\}_{n\geq 0}$ is $\frac{h(x)}{x}$. Therefore,

$$F(x) = g(x) + \frac{h(x)}{x} = \frac{1 - 2x - x^2}{(1-x)(1-4x-x^2)}.$$

For the right side sequence, we use the following result [1, equation (4.18)]:

$$\sum_{n=0}^{\infty} F_{kn+r} x^n = \frac{F_r + (-1)^r F_{k-r} x}{1 - L_k x + (-1)^k x^2}.$$

So, the generating function of $\left\{F_{3n+2}\right\}_{n\geq 0}$ is $\frac{1+x}{1-4x-x^2}$, and therefore, the generating function corresponding to the right side is

$$G(x) = \frac{1}{2}\left(\frac{1+x}{1-4x-x^2} + \frac{1}{1-x}\right) = \frac{1-2x-x^2}{(1-x)(1-4x-x^2)}.$$

Since $F(x) = G(x)$, the conclusion follows.

Editor's Notes: Several solvers used the formulas

$$\sum_{k=1}^{n} F_k^3 = \frac{F_{3n+2} + 6(-1)^{n+1}F_{n-1} + 5}{10} \quad \text{or} \quad \sum_{k=1}^{n} F_k^3 = \frac{F_{n+2}^3 - 3F_{n+1}^3 + 3(-1)^n F_n + 2}{4}$$

to complete their proofs. The equality $F_{n+1}^3 + F_n^2 - F_{n-1}^3 = F_{3n}$ can be generalized to [2, Identity 45, page 89]

$$F_{r+1}F_{s+1}F_{t+1} + F_r F_s F_t - F_{r-1}F_{s-1}F_{t-1} = F_{r+s+t}.$$

References

[1] V. E. Hoggatt Jr, and D. A. Lind, *A primer for the Fibonacci numbers: Part VI*, The Fibonacci Quarterly, **5.5** (1967) 445–460.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

Also solved by Paola Andrea Velásquez Barrientos (student), Brian D. Beasley, Bruno Berselli, Brian Bradie, Charles K. Cook, Kenny B. Davenport, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Wei-Kai Lai and John Risher (student) (jointly), Kathleen E. Lewis, Carlos Alexander Montoya Sanchez (student), Raphael Schumacher (student), Dong-chan Shin (student), Jason L. Smith, Albert Stadler, David Terr, and the proposer.

A Cyclic Sum in Disguise

B-1212 Proposed by D. M. Bătineţu-Giurgui, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 55.3, August 2017)

Prove that

$$\frac{F_n^4+1}{F_n^2-F_n+1}+\sum_{k=1}^{n-1}\frac{F_k^4+F_{k+1}^4}{F_k^2-F_kF_{k+1}+F_{k+1}^2}>2F_nF_{n+1}$$

for any positive integer n.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Let x and y be positive integers. Then,

$$x^2-xy+y^2=\left(x-\frac{1}{2}\,y\right)^2+\frac{3}{4}\,y^2>0,$$

and

$$\frac{x^4+y^4}{x^2-xy+y^2}-(x^2+y^2)=\frac{xy(x-y)^2}{x^2-xy+y^2}\geq 0,$$

so that

$$\frac{x^4+y^4}{x^2-xy+y^2}\geq x^2+y^2,$$

with equality holding if and only if $x=y$. Using this inequality and $F_1=1$, it follows that

$$\begin{aligned}
\frac{F_n^4+1}{F_n^2-F_n+1}+\sum_{k=1}^{n-1}\frac{F_k^4+F_{k+1}^4}{F_k^2-F_kF_{k+1}+F_{k+1}^2} &= \frac{F_n^4+F_1^4}{F_n^2-F_nF_1+F_1^2}+\sum_{k=1}^{n-1}\frac{F_k^4+F_{k+1}^4}{F_k^2-F_kF_{k+1}+F_{k+1}^2}\\
&\geq F_n^2+F_1^2+\sum_{k=1}^{n-1}(F_k^2+F_{k+1}^2)\\
&= 2\sum_{k=1}^{n}F_k^2\\
&= 2F_nF_{n+1}.
\end{aligned}$$

Equality holds for $n=1$ (as $F_1=F_1=1$), and for $n=2$ (as $F_1=F_2=1$), but the inequality is strict for $n>2$.

Editor's Notes: Plaza observed that the result can be extended to $n \geq 2$ distinct positive real numbers:

$$\sum_{\substack{k=1 \\ \text{cyclic}}}^{n} \frac{a_k^4 + a_{k+1}^4}{a_k^2 - a_k a_{k+1} + a_{k+1}^2} > 2 \sum_{k=1}^{n} a_k^2.$$

Also solved by Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Wei-Kai Lai and John Risher (student) (jointly), Hideyuki Ohtsuka, Ángel Plaza, and the proposer.

An Intriguing Telescoping Product

B-1213 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine. (Vol. 55.3, August 2017)

For every positive integer n, prove that

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \sqrt[4]{\frac{1}{F_1 + F_5 + \cdots + F_{8n+1}}},$$

and

$$\frac{F_2}{F_4} \cdot \frac{F_6}{F_8} \cdot \cdots \cdot \frac{F_{4n-2}}{F_{4n}} < \sqrt[4]{\frac{2}{F_3 + F_7 + \cdots + F_{8n+3}}}.$$

Solution by the proposer.

If $k < m$, then for every positive integer p, we find, by means of Binet's formula, or by applying Identity 2 in [1, page 87],

$$F_k F_{m+p} - F_{k+p} F_m = (-1)^{k+1} F_p F_{m-k}.$$

Hence, $\frac{F_k}{F_m} > \frac{F_{k+p}}{F_{m+p}}$ if k is odd, and $\frac{F_k}{F_m} < \frac{F_{k+p}}{F_{m+p}}$ if k is even. Thus,

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \frac{F_2}{F_4} \cdot \frac{F_6}{F_8} \cdot \cdots \cdot \frac{F_{4n-2}}{F_{4n}},$$

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \frac{F_3}{F_5} \cdot \frac{F_7}{F_9} \cdot \cdots \cdot \frac{F_{4n-1}}{F_{4n+1}},$$

$$\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} > \frac{F_4}{F_6} \cdot \frac{F_8}{F_{10}} \cdot \cdots \cdot \frac{F_{4n}}{F_{4n+2}}.$$

Therefore,

$$\left(\frac{F_1}{F_3} \cdot \frac{F_5}{F_7} \cdot \cdots \cdot \frac{F_{4n-3}}{F_{4n-1}} \right)^4 > \frac{F_1 F_2}{F_{4n+1} F_{4n+2}} = \frac{1}{\sum_{k=1}^{4n+1} F_k^2}.$$

The first inequality follows from

$$\sum_{k=1}^{4n+1} F_k^2 = F_1^2 + (F_2^2 + F_3^2) + \cdots + (F_{4n}^2 + F_{4n+1}^2) = F_1 + F_5 + \cdots + F_{8n+1}.$$

The second inequality can be obtained in a similar manner.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

Also solved by Kenny B. Davenport (two solutions), and Dmitry Fleischman.

Telescoping Sum Again!

B-1214 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 55.3, August 2017)

Given an integer $m \geq 2$, find a closed form for the infinite sum

$$\sum_{n=1}^{\infty} \frac{F_{2n+m}}{F_n F_{n+2} F_{n+m-2} F_{n+m}}.$$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

Consider the numerator in the summand. Using the identity [1, page 89, identity 47] $F_p F_{q+r} = F_{p+q} F_r - (-1)^p F_q F_{r-p}$, we find

$$F_{2n+m} = F_2 F_{n+(n+m)} = F_{n+2} F_{n+m} - F_n F_{n+m-2}.$$

The sum now becomes

$$\sum_{n=1}^{\infty} \left(\frac{1}{F_n F_{n+m-2}} - \frac{1}{F_{n+2} F_{n+m}} \right).$$

This is a telescoping sum in which only the first two positive terms survive. Therefore, the closed form that is sought is

$$\frac{1}{F_1 F_{m-1}} + \frac{1}{F_2 F_m} = \frac{1}{F_{m-1}} + \frac{1}{F_m} = \frac{F_m + F_{m-1}}{F_{m-1} F_m} = \frac{F_{m+1}}{F_{m-1} F_m}.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

Also solved by Brian Bradie, I. V. Fedak, Ángel Plaza, Raphael Schumacher, and the proposer.

An Infinite Series of Arctangent

B-1215 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain. (Vol. 55.3, August 2017)

For any positive integer k, the k-Fibonacci and k-Lucas sequences $\{F_{k,n}\}_{n \in \mathbb{N}}$ and $\{L_{k,n}\}_{n \in \mathbb{N}}$ are defined recursively by $u_{n+1} = k u_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k=0} = 2$, $L_{k,1} = k$. Let c be a positive integer. The sequence $\{a_n\}_{n \in \mathbb{N}}$ is defined by $a_1 = 1$, $a_2 = 3$, and $a_{n+2} = a_n + 2c$ for $n \geq 1$. Prove that

$$\sum_{n=1}^{\infty} \tan^{-1} \left(\frac{F_{k,c}}{F_{k,a_n+c}} \right) = \tan^{-1} \left(\frac{1}{k} \right), \qquad \text{if } c \text{ is even;}$$

$$\sum_{n=1}^{\infty} \tan^{-1} \left(\frac{L_{k,c}}{L_{k,a_n+c}} \right) = \tan^{-1} \left(\frac{1}{k} \right), \qquad \text{if } c \text{ is odd.}$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

We have the Binet's forms $F_{k,n} = \frac{p^n-q^n}{p-q}$, and $L_{k,n} = p^n + q^n$, with $p = \frac{k+\sqrt{k^2+4}}{2}$ and $q = \frac{k-\sqrt{k^2+4}}{2}$. Note that $pq = -1$, hence $p^{-1} = -q$. We find

$$\begin{aligned}\tan^{-1} p^{-a_n} - \tan^{-1} p^{-a_{n+2}} &= \tan^{-1} \frac{p^{-a_n} - p^{-a_{n+2}}}{1 + p^{-a_n}p^{-a_{n+2}}} \\ &= \tan^{-1} \frac{p^{(a_{n+2}-a_n)/2} - p^{-(a_{n+2}-a_n)/2}}{p^{(a_{n+2}+a_n)/2} + p^{-(a_{n+2}+a_n)/2}} \\ &= \tan^{-1} \frac{p^c - p^{-c}}{p^{a_n+c} + p^{-a_n-c}}.\end{aligned}$$

By the above inequality, for $m \geq 1$, we have

$$\begin{aligned}\sum_{n=1}^{m} \tan^{-1} \frac{p^c - p^{-c}}{p^{a_n+c} + p^{-a_n-c}} &= \sum_{n=1}^{m} \left(\tan^{-1} p^{-a_n} - \tan^{-1} p^{-a_{n+2}}\right) \\ &= \tan^{-1} p^{-a_1} + \tan^{-1} p^{-a_2} - \tan^{-1} p^{-a_{m+1}} - \tan^{-1} p^{-a_{m+2}}.\end{aligned}$$

Since

$$\tan^{-1} p^{-a_1} + \tan^{-1} p^{-a_2} = \tan^{-1} p^{-1} + \tan^{-1} p^{-3} = \tan^{-1} \frac{p^{-1} + p^{-3}}{1 - p^{-1}p^{-3}}$$

$$= \tan^{-1} \frac{p + p^{-1}}{p^2 - p^{-2}} = \tan^{-1} \frac{p - q}{p^2 - q^2} = \tan^{-1} \frac{1}{F_{k,2}} = \tan^{-1} \frac{1}{k},$$

and $\lim_{n\to\infty} a_n = \infty$, we obtain the identity

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{p^c - p^{-c}}{p^{a_n+c} + p^{-a_n-c}} = \tan^{-1} \frac{1}{k}.$$

If c is even, then $a_n + c$ is odd for $n \geq 1$, and

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{F_{k,c}}{F_{k,a_n+c}} = \sum_{n=1}^{\infty} \tan^{-1} \frac{p^c - q^c}{p^{a_n+c} - q^{a_n+c}} = \sum_{n=1}^{\infty} \tan^{-1} \frac{p^c - p^{-c}}{p^{a_n+c} + p^{-a_n-c}} = \tan^{-1} \frac{1}{k}.$$

If c is odd, then $a_n + c$ is even for $n \geq 1$, and

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{L_{k,c}}{L_{k,a_n+c}} = \sum_{n=1}^{\infty} \tan^{-1} \frac{p^c + q^c}{p^{a_n+c} + q^{a_n+c}} = \sum_{n=1}^{\infty} \tan^{-1} \frac{p^c - p^{-c}}{p^{a_n+c} + p^{-a_n-c}} = \tan^{-1} \frac{1}{k}.$$

Editor's Note: This problem is similar to Problem B-1198.

Also solved by Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, and the proposer.