ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2020. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1251 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

For any positive integers m and n, prove that

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} F_{m+k}(x+1)^k &= \sum_{k=0}^{n} F_{m+2n-k} x^k, \\ \sum_{k=0}^{n} \binom{n}{k} L_{m+k}(x+1)^k &= \sum_{k=0}^{n} L_{m+2n-k} x^k. \end{aligned}$$

B-1252 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For integers $n \geq 0$ and $r \geq 0$, prove that

$$F_{n-r}F_nF_{n+r} + \sum_{k=1}^{n+1} F_{k-r}F_kF_{k+r} = \frac{F_{3n+2}+1}{2} + (-1)^r F_r^2.$$

B-1253 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that

$$\sin F_{2n+2} + \sin F_n^2 + \cos F_{n+2}^2 \leq \frac{3}{2}.$$

B-1254 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Let B_n be the nth balancing number defined as $B_0 = 0$, $B_1 = 1$, and $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$. Show that

$$B_n + 4\sum_{k=0}^{n} kB_k \equiv 0 \pmod{n}$$

for all positive integers n.

B-1255 Proposed by José Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

For any positive integer n, prove that:

$$\sqrt{\left(\frac{F_n - 1}{F_n}\right)^{F_n}\left(\frac{L_n - 1}{L_n}\right)^{L_n}} \leq \left(\frac{F_{n+1} - 1}{F_{n+1}}\right)^{F_{n+1}}.$$

SOLUTIONS

Two Fibonacci-Lucas Identities with Central Binomial Coefficient

B-1231 Proposed by Kenny B. Davenport, Dallas, PA. (Vol. 56.3, August 2018)

Find the closed form expressions for the sums

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{nF_n}{8^n}, \qquad \text{and} \qquad \sum_{n=1}^{\infty}\binom{2n}{n}\frac{nL_n}{8^n}.$$

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

The generating function for the Catalan numbers is known to be

$$g(x) = \sum_{n=1}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

From here it follows that

$$h(x) = \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{d}{dx}\left[x\, g(x)\right] = \frac{d}{dx}\left(\frac{1-\sqrt{1-4x}}{2}\right) = \frac{1}{\sqrt{1-4x}}.$$

Therefore,

$$f(x) = \sum_{n=0}^{\infty}\binom{2n}{n}nx^n = x\, h'(x) = \frac{2x}{\sqrt{(1-4x)^3}}.$$

Thus, using Binet's formulas, we find

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nF_n}{8^n}=\frac{1}{\sqrt{5}}\left[f\left(\frac{\alpha}{8}\right)-f\left(\frac{\beta}{8}\right)\right]=\frac{1}{\sqrt{10}}\left[\frac{\alpha}{\sqrt{(2-\alpha)^3}}-\frac{\beta}{\sqrt{(2-\beta)^3}}\right].$$

Finally, use the identities $2-\alpha=\beta^2$ and $2-\beta=\alpha^2$ to obtain (recall that $\beta<0$)

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nF_n}{8^n}=\frac{1}{\sqrt{10}}\left(\frac{\alpha}{-\beta^3}-\frac{\beta}{\alpha^3}\right)=\frac{\alpha^4+\beta^4}{\sqrt{10}}=\frac{L_4}{\sqrt{10}}=\frac{7}{\sqrt{10}}.$$

Similarly,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nL_n}{8^n}=f\left(\frac{\alpha}{8}\right)+f\left(\frac{\beta}{8}\right)=\frac{1}{\sqrt{2}}\left(\frac{\alpha}{-\beta^3}+\frac{\beta}{\alpha^3}\right)=\frac{\alpha^4-\beta^4}{\sqrt{2}}=\frac{5F_4}{\sqrt{10}}=\frac{15}{\sqrt{10}}.$$

Editor's Notes: Gruebel observed that the same argument can be extended to obtain the generalization

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nF_{n+m}}{8^n}=\frac{L_{m+4}}{\sqrt{10}},\qquad\text{and}\qquad\sum_{n=0}^{\infty}\binom{2n}{n}\frac{nL_{n+m}}{8^n}=\frac{5F_{m+4}}{\sqrt{10}}.$$

Frontczak studied the relationship

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{n}=\int_0^x\sum_{n=1}^{\infty}\binom{2n}{n}t^{n-1}\,dt=\int_0^x\left(\frac{1}{\sqrt{1-4t}}-1\right)\frac{dt}{t}.$$

Applying the identity $t\,[g(t)]^2=g(t)-1$, he was able to derive the identity

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{n}=2\int_0^x\frac{g(t)}{1-2t\,g(t)}\,dt=2\int_0^x\frac{g'(t)}{g(t)}\,dt=2\ln[g(x)];$$

and consequently obtained

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{F_n}{n\,8^n}=\frac{2}{\sqrt{5}}\ln\left(\frac{\sqrt{2}+\alpha}{\sqrt{2}+\beta}\right).$$

Also solved by Michel Bataille, Khristo N. Boyadzhiev, Robert Frontczak, G. C. Gruebel, Ángel Plaza, Kevin Darío López Rodriguez and Santiago Alzate Suárez (both students) (jointly), Raphael Schumacher (student), Jason L. Smith, Albert Stadler, and the proposer.

An Application of Kantorovich's Inequality

B-1232 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 56.3, August 2018)

Let $e_n=\left(1+\frac{1}{n}\right)^n$. Prove that

$$\left(\sum_{i=1}^{n}e_iF_i^2\right)\left(\sum_{j=1}^{n}\frac{F_j^2}{e_j}\right)\le\frac{(e+2)^2}{8e}\,F_n^2F_{n+1}^2,$$

for any positive integer n.

Solution by Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.

The Kantorovich's inequality states that if $a, b \in \mathbb{R}^*$, $a < b$, and $x_i \in [a, b]$, $t_i \in \mathbb{R}^+$, then

$$\left(\sum_{i=1}^{n} t_i x_i\right)\left(\sum_{j=1}^{n} \frac{t_i}{x_i}\right) \le \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^{n} t_i\right).$$

Putting $t_i = F_i^2$, and taking into account the famous inequality $2 < e_i = \left(1 + \frac{1}{i}\right)^i \le e$, we have

$$\left(\sum_{i=1}^{n} e_i F_i^2\right)\left(\sum_{i=1}^{n} \frac{F_i^2}{e_i}\right) \le \frac{(e+2)^2}{8e}\left(\sum_{i=1}^{n} F_i^2\right) \le \frac{(e+2)^2}{8e} F_n^2 F_{n+1}^2.$$

Editor's Note: Plaza showed that

$$\left(\sum_{i=1}^{n} e_i L_i^2\right)\left(\sum_{j=1}^{n} \frac{L_j^2}{e_j}\right) \le \frac{(e+2)^2}{8e}\left(L_n L_{n+2} - 2\right)^2.$$

Using an elaborate analysis, Stadler managed to obtain a slightly stronger result:

$$\left(\sum_{i=1}^{n} e_i F_i^2\right)\left(\sum_{j=1}^{n} \frac{F_j^2}{e_j}\right) \le \frac{562057}{559872} F_n^2 F_{n+2}^2.$$

Also solved by I. V. Fedak, Dmitry Fleischman, Ángel Plaza, Kevin Darío López Rodríguez (student), Albert Stadler, Anthony Vasaturo, and the proposer.

A Recursive Technique

B-1233 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 56.3, August 2018)

For any positive integer n, prove that

$$\sum_{k=1}^{n} F_{2k^2} F_{2k} = F_{n(n+1)}^2, \qquad \text{and} \qquad \sum_{k=1}^{n} F_{2F_k^2} F_{2F_{2k}} = F_{2F_n F_{n+1}}^2.$$

Solution 1 by Thomas Koshy (retired), Framingham Sate University, Framingham, MA.

Proofs of both parts employ a recursive technique [1, 2]. For the first identity, let $A_n = \sum_{k=1}^{n} F_{2k^2} F_{2k}$ and $B_n = F_{n(n+1)}^2$. Using the identity [2]

$$F_{a+b}^2 - F_{a-b}^2 = F_{2a} F_{2b}, \tag{1}$$

we have

$$B_n - B_{n-1} = F_{n^2+n} - F_{n^2-n} = F_{2n^2} F_{2n} = A_n - A_{n-1}.$$

Hence,

$$A_n - B_n = A_{n-1} - B_{n-1} = \cdots = A_1 - B_1 = 0.$$

Consequently, $A_n = B_n$, as desired.

For the second identity, let $C_n = \sum_{k=1}^n F_{2F_k^2} F_{2F_{2k}}$ and $D_n = F^2_{2F_n F_{n+1}}$. Since $F_{2n} = F_n L_n$, and $F_n + L_n = 2F_{n+1}$, we find

$$2F_n F_{n+1} = F_n(F_n + L_n) = F_n^2 + F_{2n},$$

and

$$2F_{n-1}F_n = (2F_{n+1} - 2F_n)F_n = (L_n - F_n)F_n = F_{2n} - F_n^2.$$

By identity (1), we have

$$D_n - D_{n-1} = F^2_{F_{2n}+F_n^2} - F^2_{F_{2n}-F_n^2} = F_{2F_{2n}} F_{2F_n^2} = C_n - C_{n-1}.$$

This implies

$$C_n - D_n = C_{n-1} - D_{n-1} = \cdots = C_1 - D_1 = 0;$$

so $C_n = D_n$, as desired.

Solution 2 by Robert Frontczak, Landesbank Baden-Wüttemberg, Stuttgart, Germany.

Both identities are special instances of the following general result:

Proposition: *Let $f(n)$ and $g(n)$ be two integer-valued functions. Consider the composition $f(g(n))$. Assume that $f(g(0)) \geq 0$, and $f(g(n))$ is increasing for $n \geq 0$. If $f(g(n)) + f(g(n-1))$ is even for all integers n, then*

$$\sum_{k=1}^{n} F_{f(g(k))+f(g(k-1))} \cdot F_{f(g(k))-f(g(k-1))} = F^2_{f(g(n))} - F^2_{f(g(0))}.$$

Proof of the Proposition: Using Catalan's Identity $F_{s+t} \cdot F_{s-t} = F_s^2 + (-1)^{s+t+1} F_t^2$ with $s = f(g(k))$ and $t = f(g(k-1)$, we find

$$F_{f(g(k))+f(g(k-1))} \cdot F_{f(g(k))-f(g(k-1))} = F^2_{f(g(k))} - F^2_{f(g(k-1))}.$$

Therefore, the sum in the proposition is telescopic, and the result follows immediately. □

To obtain the first identity, set $f(n) = n$ and $g(n) = n(n+1)$, and observe that for $k = 1, 2, \ldots, n$,

$$\begin{aligned} f(g(k)) + f(g(k-1)) &= 2k^2, \\ f(g(k)) - f(g(k-1)) &= 2k. \end{aligned}$$

The second identity is obtained by setting $f(n) = 2n$ and $g(n) = F_n F_{n+1}$. In this case,

$$f(g(k)) + f(g(k-1)) = 2F_k F_{k+1} + 2F_k F_{k-1} = 2F_k L_k = 2F_{2k},$$

and

$$f(g(k)) - f(g(k-1)) = 2F_k F_{k+1} - 2F_k F_{k-1} = 2F_k^2.$$

Setting $f(n) = 2n$ and $g(n) = L_{n+1}$ produces another example involving Lucas numbers:

$$\sum_{k=1}^{n} F_{2L_{k+2}} \cdot F_{2L_{k-1}} = F^2_{2L_{n+1}} - 1.$$

Editor's Note: Greubel studied the general recurrence

$$\phi_0 = 0, \quad \phi_1 = 1, \qquad \phi_{n+2} = a\,\phi_{n+1} + b\,\phi_n, \quad n \geq 0,$$

and the summation $\sum_{k=0}^n \phi_{2k^2}\phi_{2k}$. Although no simple closed form was obtained, he was able to derive the interesting results (where P_n is the nth Pell number)

$$\sum_{k=0}^{n} F_{6k^2}F_{6k} = F^2_{3n(n+1)}, \qquad \sum_{k=0}^{n} F_{4k^2}F_{4k} = F^2_{2n(n+1)}, \qquad \sum_{k=0}^{n} P_{2k^2}P_{2k} = P^2_{n(n+1)}.$$

References

[1] S. Clary ad P. D. Hemenway, On sums of cubes of Fibonacci numbers, *Applications of Fibonacci Numbers, Vol. 5* (ed. G. E. Bergum, et. al.), 1993, Kluwer, Dordrecht, 1993, 123–136.

[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications, Vol. II*, John Wiley, NY, 2019.

Also solved by I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Wei-Kai Lai, Ehren Metcalfe, Ángel Plaza, Ralphael Schumacher (student), Albert Stadler, Santiago Alzate Suárez (student), Anthony Vasaturo, and the proposer.

A Convoluted System of Equations

B-1234 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 56.3, August 2018)

Let $n \geq 3$ be an odd integer. Find the real solutions of the following system of equations:

$$\begin{aligned} x_1^3 + x_1 + x_2 &= F_1 x_1^2 + F_3, \\ x_2^5 + x_2 + x_3 &= F_2 x_2^4 + F_4, \\ &\vdots \\ x_{n-1}^{2n-1} + x_{n-1} + x_n &= F_{n-1} x_{n-1}^{2n-2} + F_{n+1}, \\ x_n^{2n+1} + \frac{F_{n+2}-1}{F_n} x_n + x_1 &= F_n x_n^{2n} + F_{n+2}. \end{aligned}$$

Solution by the Proposer.

We use $F_{k+2} = F_{k+1} + F_k$, and write the system of equation as

$$\begin{aligned} (x_1 - F_1)(x_1^2 + 1) &= F_2 - x_2, \\ (x_2 - F_2)(x_2^4 + 1) &= F_3 - x_3, \\ &\vdots \\ (x_{n-1} - F_{n-1})\big(x_{n-1}^{2n-2} + 1\big) &= F_{n-1} - x_{n-1}, \\ (x_n - F_n)\left(x_n^{2n} + \frac{F_{n+2}-1}{F_n}\right) &= F_1 - x_1. \end{aligned}$$

If $x_1 > F_1$, then $x_2 < F_2$; thus, $x_3 > F_3$, etc. Since n is odd, this ends with $x_n > F_n$, which in turn implies that $x_1 < F_1$. This contradiction asserts that $x_1 \leq F_1$. Likewise, we also have $x_1 \geq F_1$. Therefore, $x_1 = F_1$. Consequently, $x_k = F_k$ for $k = 1, 2, \ldots, n$.

Editor's Note: Plaza extended the result to a Lucas analog. The equations are similar, except that the Fibonacci numbers are replaced by the Lucas numbers. The unique solution is $x_k = L_k$ for $k = 1, 2, \ldots, n$.

Also solved by Charles K. Cook, Dmitry Fleischman, Ángel Plaza, Kevin Darío López Rodríguez (student), and the proposer.

Solution From an Old Problem

B-1235 Proposed by Kenny B. Davenport, Dallas, PA.
(Vol. 56.3, August 2018)

Prove that, for any integer $n \geq 1$,

$$\sum_{k=1}^{n} F_{k-1}F_kF_{k+1} = \frac{1}{3}\left(F_{n-1}^3 + F_n^3 + F_{n+1}^3 - \frac{F_{3n-1}+3}{2}\right).$$

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

We use Elementary Problem B-1211 from this *Quarterly* (Volume 56, Number 3), which states that

$$\sum_{k=1}^{n} F_k^3 = \frac{F_{3n-1}+1}{2} - F_{n-1}^3,$$

and the identity [1, Identity 90, page 91]

$$3F_{k+1}F_kF_{k-1} = F_{k+1}^3 - F_k^3 - F_{k-1}^3 \tag{2}$$

We find

$$\sum_{k=1}^{n} F_{k-1}F_kF_{k+1} = \frac{1}{3}\left[\sum_{k=1}^{n}\left(F_{k+1}^3 - F_{k-1}^3\right) - \sum_{k=1}^{n} F_k^3\right].$$

The first sum on the right side is telescopic, and using Problem B-1211 on the second sum, we have

$$\begin{aligned}\sum_{k=1}^{n} F_{k-1}F_kF_{k+1} &= \frac{1}{3}\left[F_{n+1}^3 + F_n^3 - F_1^3 - \left(\frac{F_{3n-1}+1}{2} - F_{n-1}^3\right)\right] \\ &= \frac{1}{3}\left(F_{n-1}^3 + F_n^3 + F_{n+1}^3 - \frac{F_{3n-1}+3}{2}\right).\end{aligned}$$

Editor's Note: Metcalfe used a computer algorithm to establish the identity. Continuing with the given identity, Ohtsuka showed that its right side can be further simplified:

$$\sum_{k=1}^{n} F_{k-1}F_kF_{k+1} = \frac{4F_{n-1}^3 + L_{3n-1} - 3}{6}.$$

This problem appeared in [2] with a different closed form on the right side.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, NY, 2001.
[2] Z. F. Starc, Solution to Problem 44, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, **14** (2013), 113.

Also solved by Michel Bataille, Brian D. Beasley, Pridon Davlianidze, Steve Edwards (two other solutions), I. V. Fedak, Dmitry Fleischman, Robert Frontczak, G. C, Greubel, Ralph P. Grimaldi, Thomas Koshy, Wei-Kai Lai, Ehren Metcalfe, Hideyuki Ohtsuka, Ángel Plaza, Keven Darío López Rodríguez (student), Raphael Schumacher (student), Jason L. Smith, Albert Stadler, Santiago Alzate Suárez (student), Anthony Vasaturo, and the proposer.