ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2021. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-415 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.
(Vol. 17.4, December 1979)

The circumference of a circle in a fixed plane is partitioned into n arcs of equal length. In how many ways can one color these arcs if each arc must be red, white, or blue? Colorings that can be rotated into one another should be considered to be the same.

Editor's Note: This is another old problem from 40 years ago. No solutions have appeared, so we feature the problem again, and invite the readers to solve it.

B-1271 Proposed by Ivan V. Fedak, Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n, prove that

$$\frac{\alpha^n - \beta^n}{\alpha^{n+3} - \beta^{n+3}} + \frac{\alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)(\alpha^{n+1} + \beta^{n+1})} > \frac{1}{2}.$$

B-1272 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any integer $n \geq 0$, prove that

(i) $\displaystyle\sum_{k=0}^{n} \beta^k \binom{n}{k} \cos\frac{k\pi}{5} = (-\beta)^n \cos\frac{n\pi}{5}$,

(ii) $\displaystyle\sum_{k=0}^{n} \beta^k \binom{n}{k} \sin\frac{k\pi}{5} = -(-\beta)^n \sin\frac{n\pi}{5}$.

B-1273 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Let $\{u_n\}_{n\geq 0}$ be a generalized Fibonacci sequence defined by $u_n = u_{n-1} + u_{n-2}$ with u_0 and u_1 not both being zero. Let $\{a_n\}_{n\geq 1}$ be an arithmetic progression, that is, $a_n = a_1 + (n-1)d$, where $a_1, d > 0$. Show that

$$\sum_{k=1}^{n} \frac{u_{k+2}}{a_{k+1}\sqrt{a_k} + a_k\sqrt{a_{k+1}}} = \frac{1}{d}\left(\frac{u_3}{\sqrt{a_1}} - \frac{u_{n+2}}{\sqrt{a_{n+1}}} + \sum_{k=1}^{n-1} \frac{u_{k+1}}{\sqrt{a_{k+1}}}\right).$$

B-1274 Proposed by Ivan V. Fedak, Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n, prove that

$$\sum_{k=1}^{n} \sqrt{F_{2k-1}} \geq \sqrt{F_{2n+3} - 2F_{n+3} + 2}.$$

B-1275 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Given a real number $c > 0$, for any integer $n \geq 0$, find a closed form expression for the sum

$$\sum_{k=0}^{n} \prod_{j=k}^{n} \frac{1}{c(L_{2^{j+1}} + 1) + L_{2^j} - 1}.$$

SOLUTIONS

A Binomial Sum of Generalized Fibonacci Numbers

B-1251 (Corrected) Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.
(Vol. 57.3, August 2019)

For any positive integers m and n, prove that

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} F_{m+k}(x+1)^k &= \sum_{k=0}^{n} \binom{n}{k} F_{m+2n-k} x^k,\\ \sum_{k=0}^{n} \binom{n}{k} L_{m+k}(x+1)^k &= \sum_{k=0}^{n} \binom{n}{k} L_{m+2n-k} x^k.\end{aligned}$$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

Let γ represent either α or β. Consider the sum

$$\sum_{k=1}^{n}\binom{n}{k}\gamma^k(x+1)^k=[1+\gamma(x+1)]^n=(\gamma+1+\gamma x)^n.$$

Since $\gamma^2=\gamma+1$, we obtain

$$(\gamma+1+\gamma x)^n=(\gamma^2+\gamma x)^n=\gamma^n(\gamma+x)^n=\gamma^n\sum_{k=0}^{n}\binom{n}{k}\gamma^{n-k}x^k.$$

Therefore,

$$\sum_{k=1}^{n}\binom{n}{k}\gamma^{m+k}(x+1)^k=\sum_{k=0}^{n}\binom{n}{k}\gamma^{m+2n-k}x^k.$$

Because this identity is satisfied using $\gamma=\alpha$ or $\gamma=\beta$, it is also satisfied by any generalized Fibonacci sequence expressible as $G_r=a\alpha^r+b\beta^r$. Thus, we can claim in general that

$$\sum_{k=0}^{n}\binom{n}{k}G_{m+k}(x+1)^k=\sum_{k=0}^{n}\binom{n}{k}G_{m+2n-k}x^k,$$

which proves both identities.

Editor's Notes: Based on Frontczak's observation, it is easy to further extend the result to $\sum_{k=0}\binom{n}{k}G_{m+k}(x+p)^k=\sum_{k=0}^{n}\binom{n}{k}G_{m+2n-k}(x+p-1)^k$ for any real number p.

Also solved by Ulrich Abel, Michel Bataille, Khristo N. Boyadzhiev, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Raphael Schumacher (student), Albert Stadler, David Terr, and the proposer.

Catalan and an Old Elementary Problem

B-1252 **Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 57.3, August 2019)**

For integers $n\geq 0$ and $r\geq 0$, prove that

$$F_{n-r}F_nF_{n+r}+\sum_{k=1}^{n+1}F_{k-r}F_kF_{k+r}=\frac{F_{3n+2}+1}{2}+(-1)^rF_r^2.$$

Solution by Steve Edwards, Roswell, GA.

We use two identities from [1]: the Catalan identity (page 106)

$$F_{k-r}F_{k+r}=F_k^2+(-1)^{k+r+1}F_r^2,$$

and

$$\sum_{k=1}^{n+1}(-1)^{k+1}F_k=1+(-1)^nF_n.$$

We also use the identity from Problem B-1211 from this *Quarterly* [4]

$$\sum_{k=1}^{n+1}F_k^3=\frac{F_{3n+2}+1}{2}-F_n^3.$$

We have

$$\begin{aligned}
\sum_{k=1}^{n+1} F_{k-r}F_kF_{k+r} &= \sum_{k=1}^{n+1} F_k\left[F_k^2+(-1)^{k+r+1}F_r^2\right] \\
&= \sum_{k=1}^{n+1} F_k^3+(-1)^r F_r^2\sum_{k=1}^{n+1}(-1)^{k+1}F_k \\
&= \frac{F_{3n+2}+1}{2}-F_n^3+(-1)^rF_r^2\left[1+(-1)^nF_n\right] \\
&= \frac{F_{3n+2}+1}{2}+(-1)^rF_r^2-F_n\left[F_n^2+(-1)^{n+r+1}F_r^2\right] \\
&= \frac{F_{3n+2}+1}{2}+(-1)^rF_r^2-F_{n-r}F_nF_{n+r}.
\end{aligned}$$

This completes the proof.

Editor's Notes: Using the following identities [2, 3]

$$\begin{aligned}
L_{n+r}L_{n-r}-L_n^2 &= 5(-1)^{n+r}F_r^2, \\
2\sum_{k=0}^{n} L_k^3+2L_{n-1}^3 &= 5L_{3n-1}+19, \\
\sum_{k=1}^{n+1}(-1)^kL_k &= (-1)^{n+1}L_n+1.
\end{aligned}$$

Koshy derived the Lucas analog

$$L_{n-r}L_nL_{n+r}+\sum_{k=1}^{n+1}L_{k-r}L_kL_{k+r}=\frac{5L_{3n+2}+3}{2}+5(-1)^rF_r^2.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications, Vol. II*, John Wiley & Sons, New York, 2019.
[3] T. Koshy and Z. Gao, *Extended Gibonacci sums of polynomial products of order 3 revisited*, The Fibonacci Quarterly, to appear.
[4] H. Ohtsuka, *Problem B-1211*, The Fibonacci Quarterly, **55.3** (2017), 276.

Also solved by Michel Bataille, Brian Bradie, Steve Edwards (second solution), I. V. Fedak, Dmitry Fleischman, Robert Fontczak, Thomas Koshy, Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, and the proposer.

A Trigonometric Inequality

B-1253 D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 57.3, August 2019)

Prove that

$$\sin F_{2n+2}+\sin F_n^2+\cos F_{n+2}^2\le\frac{3}{2}.$$

Solution 1 by Hideyuki Ohtsuka, Saitama, Japan.

Since
$$F_{n+2}^2 - F_n^2 = (F_{n+2} - F_n)(F_{n+2} + F_n) = F_{n+1}L_{n+1} = F_{2(n+1)},$$
we obtain the identity
$$F_{2n+2} + F_n^2 = F_{n+2}^2.$$
Then,
$$\begin{aligned}
&\sin F_{2n+2} + \sin F_n^2 + \cos F_{n+2}^2 \\
&\quad = \quad 2\sin\left(\frac{F_{2n+2} + F_n^2}{2}\right)\cos\left(\frac{F_{2n+2} - F_n^2}{2}\right) + \cos\left(2\cdot\frac{F_{n+2}^2}{2}\right) \\
&\quad = \quad 2\sin\left(\frac{F_{n+2}^2}{2}\right)\cos\left(\frac{F_{2n+2} - F_n^2}{2}\right) + 1 - 2\sin^2\left(\frac{F_{n+2}^2}{2}\right).
\end{aligned}$$
Let $p = \sin\left(\frac{F_{n+2}^2}{2}\right)$, and $q = \cos\left(\frac{F_{2n+2}-F_n^2}{2}\right)$, so that
$$\begin{aligned}
&2\sin\left(\frac{F_{n+2}^2}{2}\right)\cos\left(\frac{F_{2n+2} - F_n^2}{2}\right) + 1 - 2\sin^2\left(\frac{F_{n+2}^2}{2}\right) \\
&\qquad = 2pq + 1 - 2p^2 = -2\left(p - \frac{q}{2}\right)^2 + \frac{q^2}{2} + 1 \le 0 + \frac{1}{2} + 1 = \frac{3}{2}.
\end{aligned}$$

Solution 2 by Ivan V. Fedak, Precarpathian National University, Ivano-Frankivsk, Ukraine.

Using the identity $F_{2n+2} = F_{n+2}^2 - F_n^2$, we consider the more general inequality
$$\sin(x - y) + \sin y + \cos x \le \frac{3}{2}.$$
It is equivalent to
$$\sin x\,\cos y - (1 - \sin y)(1 - \cos x) \le \frac{1}{2}.$$
This inequality is trivial when $\sin x\,\cos y \le 0$, so we may assume $\sin x\,\cos y > 0$. Let $a = \cos x$, and $b = \sin y$. Then,
$$\begin{aligned}
\sin x\,\cos y - (1 - \sin y)(1 - \cos x) \quad &= \quad \sqrt{1 - a^2}\cdot\sqrt{1 - b^2} - (1 - a)(1 - b) \\
&\le \quad \frac{(1 - a^2) + (1 - b^2)}{2} - (1 - a)(1 - b) \\
&= \quad \frac{(a + b)(2 - a - b)}{2}.
\end{aligned}$$
From $(t - 1)^2 \ge 0$, we deduce that $t(2 - t) \le 1$. The proof is completed by setting $t = a + b$.

Editor's Notes: Bradie used calculus to show that $-3 \le \sin(x - y) + \sin y + \cos x \le \frac{3}{2}$.

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Julio Cesar Mohnsam, Daniel Văcaru, and the proposer.

Reducing the Balancing Numbers Modulo n

B-1254 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.
(Vol. 57.3, August 2019)

Let B_n be the nth balancing number defined as $B_0 = 0$, $B_1 = 1$, and $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$. Show that

$$B_n + 4\sum_{k=0}^{n} kB_k \equiv 0 \pmod{n}$$

for all positive integers n.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

We will show that

$$B_n + 4\sum_{k=0}^{n} kB_k = n(B_{n+1} - B_n)$$

for all positive integers n, from which the desired congruence follows immediately. The identity is easy to verify when $n = 1$. Assume

$$B_n + 4\sum_{k=0}^{n} kB_k = n(B_{n+1} - B_n)$$

for some positive integer n. Then,

$$\begin{aligned} B_{n+1} + 4\sum_{k=0}^{n+1} kB_k &= (4n+5)B_{n+1} + 4\sum_{k=0}^{n} kB_k \\ &= (4n+5)B_{n+1} - B_n + n(B_{n+1} - B_n) \\ &= (n+1)(5B_{n+1} - B_n) \\ &= (n+1)(6B_{n+1} - B_n - B_{n+1}) \\ &= (n+1)(B_{n+2} - B_{n+1}). \end{aligned}$$

The identity follows by induction.

Editor's Notes: Davenport noticed that for the generalized balanced numbers with $B_0 = a$ and $B_1 = b$, the congruence becomes

$$B_n + 4\sum_{k=1}^{n} kB_k \equiv a \pmod{n}.$$

Going in a different direction, Fedak observed that

$$C_n + (p-2)\sum_{k=0}^{n} kC_k = n(C_{n+1} - C_n) \equiv 0 \pmod{n},$$

where $C_0 = 0$, $C_1 = 1$, and $C_n = p\,C_{n-1} - C_{n-2}$ for $n \geq 2$.

Also solved by Michel Bataille, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Ernest James (student), Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, David Terr, and the proposer.

Another Solution Using Jensen's Inequality

B-1255 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 57.3, August 2019)

For any positive integer n, prove that it holds:

$$\sqrt{\left(\frac{F_n-1}{F_n}\right)^{F_n}\left(\frac{L_n-1}{L_n}\right)^{L_n}} \leq \left(\frac{F_{n+1}-1}{F_{n+1}}\right)^{F_{n+1}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

The function $f(x) = x\log\left(1-\frac{1}{x}\right)$ is concave for $x > 1$, since $f''(x) = -\frac{1}{x(x-1)^2} < 0$. Therefore, by Jensen's inequality,

$$f(x)+f(y) \leq 2\,f\left(\frac{x+y}{2}\right).$$

Hence,

$$x\log\left(1-\frac{1}{x}\right)+y\log\left(1-\frac{1}{y}\right) \leq (x+y)\log\left(1-\frac{2}{x+y}\right)$$

for all $x, y > 1$. We put $x = F_n$, $y = L_n$. Then, $F_n + L_n = 2F_{n+1}$, and the statement follows for $n > 2$. The statement holds trivially true for $n = 1$ and $n = 2$.

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Ángel Plaza, David Terr, and the proposer.