ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2023. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1311 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let r be an integer. For any positive even integer n, prove that

$$\sum_{k=1}^{n}(-1)^k (F_{rk}F_{rk+r})^2 = \frac{(F_{rn}F_{rn+2r})^2}{L_{2r}}.$$

B-1312 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For any positive integer n, find closed form expressions for the sums

$$\sum_{k=1}^{n} L_{F_k}L_{F_{k+1}}F_{F_k}F_{F_{k+1}} \qquad \text{and} \qquad \sum_{k=1}^{n} L_{L_k}L_{L_{k+1}}F_{L_k}F_{L_{k+1}}.$$

B-1313 Proposed by Daniel Văcaru, Economic College Maria Teiuleanu, Piteşti, Romania, and Mihály Bencze, Áprily Lajos National College, Braşov, Romania.

For $a \leq -1$, show that

$$\sum_{k=1}^{n} F_k(F_{n+2} - F_k - 1)^a \geq \left(\frac{n-1}{n}\right)^a (F_{n+2} - 1)^{a+1},$$

$$\sum_{k=1}^{n} F_k^2(F_nF_{n+1} - F_k)^a \geq \left(\frac{n-1}{n}\right)^a (F_nF_{n+1})^{a+1}.$$

B-1314 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Show that

$$\sum_{k=1}^{n} \frac{F_{4k}^2 - 1}{F_{2k}^2 + 1} = F_{4n+2} - (3n+1).$$

B-1315 Proposed by Michel Bataille, Rouen, France.

For any positive integer n, let $U_n = \sum_{k=1}^{2n-1} \frac{1}{F_kF_{k+2}}$ and $V_n = \prod_{k=1}^{n} \frac{1}{2 - U_k}$. Prove that U_n is the ratio of two consecutive integers, and V_n is the ratio of two consecutive Fibonacci numbers. Lastly, evaluate $\prod_{n=1}^{\infty}\left(\sum_{k=1}^{2n} \frac{1}{F_kF_{k+2}}\right)$.

SOLUTIONS

Connection to the Inverse Hyperbolic Cotangent Function

B-1291 Proposed by Diego Rattaggi, Realgymnasium Rämibühl, Zürich, Switzerland.
(Vol. 59.3, August 2021)

Let $m \in \mathbb{N}$. Express the value of

$$\sum_{n=1}^{\infty} \frac{(\alpha^{2n-1}+1)(\alpha^{2n-1}-1)}{(2n-1)\cdot \alpha^{4m(2n-1)}}$$

in terms of Lucas numbers.

Solution by Albert Stadler, Herrliberg, Switzerland.

We note that for $|x| < 1$,

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Then

$$\sum_{n=1}^{\infty}\frac{(\alpha^{2n-1}+1)(\alpha^{2n-1}-1)}{(2n-1)\cdot\alpha^{4m(2n-1)}}=\sum_{n=1}^{\infty}\frac{\alpha^{(2-4m)(2n-1)}}{2n-1}-\sum_{n=1}^{\infty}\frac{\alpha^{-4m(2n-1)}}{2n-1}$$
$$=\frac{1}{2}\ln\left(\frac{1+\alpha^{2-4m}}{1-\alpha^{2-4m}}\right)-\frac{1}{2}\ln\left(\frac{1+\alpha^{-4m}}{1-\alpha^{-4m}}\right)=\frac{1}{2}\ln\left(\frac{(1+\alpha^{2-4m})(1-\alpha^{-4m})}{(1-\alpha^{2-4m})(1+\alpha^{-4m})}\right)$$
$$=\frac{1}{2}\ln\left(\frac{(\alpha^{2m-1}+\alpha^{1-2m})(\alpha^{2m}-\alpha^{-2m})}{(\alpha^{2m-1}-\alpha^{1-2m})(\alpha^{2m}+\alpha^{-2m})}\right)=\frac{1}{2}\ln\left(\frac{(\alpha^{2m-1}-\beta^{2m-1})(\alpha^{2m}-\beta^{2m})}{(\alpha^{2m-1}+\beta^{2m-1})(\alpha^{2m}+\beta^{2m})}\right)$$
$$=\frac{1}{2}\ln\left(\frac{\alpha^{4m-1}+\beta^{4m-1}+\alpha+\beta}{\alpha^{4m-1}+\beta^{4m-1}-\alpha-\beta}\right)=\frac{1}{2}\ln\left(\frac{L_{4m-1}+1}{L_{4m-1}-1}\right).$$

Editor's Note: Several solvers noted that the result can be expressed as $\coth^{-1}(L_{4m-1})$.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Nandan Sai Dasireddy, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Jason L. Smith, Seán M. Stewart, Andrés Ventas, and the proposer.

An Application of Nesbitt's Inequality

B-1292 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Bazău, Romania.
(Vol. 59.3, August 2021)

For $x, y, z > 0$, prove that

$$\frac{x^2}{(5F_{2n}^2y+2z)(5F_{2n}^2z+2y)}+\frac{y^2}{(5F_{2n}^2z+2x)(5F_{2n}^2x+2z)}+\frac{z^2}{(5F_{2n}^2x+2y)(5F_{2n}^2y+2x)}\geq\frac{3}{L_{4n}^2}.$$

Solution by Michel Bataille, Rouen, France.

Since $5F_{2n}^2+2=L_{4n}$, we obtain

$$\begin{aligned}2\,(5F_{2n}^2y+2z)(5F_{2n}^2z+2y) &= (25F_{2n}^4+4)\cdot 2yz+20\,(y^2+z^2)F_{2n}^2\\ &\leq (25F_{2n}^4+4)(y^2+z^2)+20\,(y^2+z^2)F_{2n}^2\\ &= (y^2+z^2)(5F_{2n}^2+2)^2\\ &= (y^2+z^2)L_{4n}^2.\end{aligned}$$

Therefore, due to symmetry, we have

$$\begin{aligned}&\frac{x^2}{(5F_{2n}^2y+2z)(5F_{2n}^2z+2y)}+\frac{y^2}{(5F_{2n}^2z+2x)(5F_{2n}^2x+2z)}+\frac{z^2}{(5F_{2n}^2x+2y)(5F_{2n}^2y+2x)}\\ &\geq\frac{2}{L_{4n}^2}\left(\frac{x^2}{y^2+z^2}+\frac{y^2}{z^2+x^2}+\frac{z^2}{x^2+y^2}\right).\end{aligned}$$

The desired result now directly follows from Nesbitt's inequality $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ for $a, b, c > 0$.

Also solved by Brian Bradie, Nandan Sai Dasireddy, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Albert Stadler, Andrés Ventas, and the proposer.

Iterated Radical of the Cubes of Fibonacci/Lucas Numbers

B-1293 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 59.3, August 2021)

For all positive integers n, prove that

(A) $\sqrt{F_1^3+\sqrt{F_2^3+\cdots+\sqrt{F_n^3}}}<2$;

(B) $\sqrt{L_1^3+\sqrt{L_2^3+\cdots+\sqrt{L_n^3}}}<3$.

Solution 1 by Hideyuki Ohtsuka, Saitama, Japan.

(A) The inequality holds when $n=1,2,3$, so we may assume $n\geq 4$. Because $2F_n-F_{n+1}=F_{n-2}\geq 1>0$, we have $2F_n>F_{n+1}$. It follows that

$$F_n^4-F_n^3-F_{n+1}^2>F_n^4-F_n^3-4F_n^2=F_n^2[F_n(F_n-1)-4]\geq F_4^2[F_4(F_4-1)-4]>0.$$

Thus, $F_n^4>F_n^3+F_{n+1}^2$, on $F_n^2>\sqrt{F_n^3+F_{n+1}^2}$. Using this inequality repeatedly, we obtain

$$2>\sqrt{F_1^3+\sqrt{F_2^3+\sqrt{F_3^3+F_4^2}}}>\sqrt{F_1^3+\sqrt{F_2^3+\sqrt{F_3^3+\sqrt{F_4^3+F_5^2}}}}$$

$$>\cdots>\sqrt{F_1^3+\sqrt{F_2^3+\sqrt{\cdots+\sqrt{F_n^3+F_{n+1}^2}}}}>\sqrt{F_1^3+\sqrt{F_2^3+\sqrt{\cdots+\sqrt{F_n^3}}}}.$$

Therefore, the desired inequality holds for any positive integer n.

(B) Similar to (A), we may assume $n\geq 3$. We find $L_n^2>\sqrt{L_n^3+L_{n+1}^2}$. Using this inequality repeatedly, we obtain

$$3>\sqrt{L_1^3+\sqrt{L_2^3+L_3^2}}>\sqrt{L_1^3+\sqrt{L_2^3+\sqrt{L_3^3+L_4^2}}}$$

$$>\cdots>\sqrt{L_1^3+\sqrt{L_2^3+\sqrt{\cdots+\sqrt{L_n^3+L_{n+1}^2}}}}>\sqrt{L_1^3+\sqrt{L_2^3+\sqrt{\cdots+\sqrt{L_n^3}}}}.$$

Therefore, the desired inequality holds for any positive integer n.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

According to the Hershfeld's Convergence Theorem [1], if $0<a_n<M^{2^n}$, then

$$\sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}}<\sqrt{M}\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}.$$

(A) Because $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = \alpha$, it is enough to prove that $F_n^3 < (4/\alpha^2)^{2^n}$, which follows because $4/\alpha^2 > 2$, and $2^{2^n} > F_n^3$.

(B) Analogously, it is enough to prove that $L_n^3 < (9/\alpha^2)^{2^n}$, which follows because $9/\alpha^2 > 3$, and $3^{2^n} > L_n^3$.

Reference

[1] A. Hershfeld, *On infinite radicals*, Amer. Math. Monthly, **42** (1935), 419–429.

Also solved by Michel Bataille, Dmitry Fleischman, Albert Stadler, Andrés Ventas, and the proposer.

Lagrange Interpolation Formula

B-1294 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Gran Canaria, Spain, and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.
(Vol. 59.3, August 2021)

Let $A(x)$ and $B(x)$ be polynomials of degree n such that $A(i) = F_i$ and $B(i) = L_i$, respectively, for every i with $0 \le i \le n$. Find the values of $A(n+1)$ and $B(n+1)$.

Solution by Ernest James (undergraduate student), the Citadel, Charleston, SC.

(A) Using the Lagrange interpolation formula, we find

$$A(x) = \sum_{k=0}^{n} F_k \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{x-j}{k-j}.$$

Now we see that

$$A(n+1) = \sum_{k=0}^{n} F_k \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{n+1-j}{k-j} = \sum_{k=0}^{n} \frac{(-1)^{n-k}(n+1)}{n-k+1}\binom{n}{k} F_k.$$

By combining Binet's formula and the binomial theorem, we have

$$\sum_{k=0}^{n} \binom{n}{k} F_k (-x)^{n-k} = \frac{1}{\sqrt{5}}\left[(\alpha - x)^n - (\beta - x)^n\right].$$

Next, multiply both sides by $n+1$ and integrate over $0 \le x \le 1$:

$$\int_0^1 \sum_{k=0}^{n} (n+1)\binom{n}{k} F_k(-x)^{n-k}\, dx = \int_0^1 \frac{n+1}{\sqrt{5}}\left[(\alpha - x)^n - (\beta - x)^n\right] dx.$$

Doing so, we obtain

$$A(n+1) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(n+1)}{n-k+1}\binom{n}{k} F_k = \left[1 - (-1)^n\right] F_{n+1}.$$

(B) We can see, similarly to what we did above, that $B(n+1) = \left[1 + (-1)^n\right] L_{n+1}$.

Editor's Note: Frontczak remarked that a general result can be found in [1].

Reference

[1] A. M. Alt, *Numerical sequences and polynomials*, Arhimede Mathematical Journal, **2** (2019), 114–120.

Also solved by Ulrich Abel, Michael R. Bacon and Charles K. Cook (jointly), Michel Bataille, Brian Bradie, Dmitry Fleischman, Robert Frontczak, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.

A Sum with Reciprocals of the Central Binomial Coefficients

B-1295 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 59.3, August 2021)

Given an even integer r, prove that

$$\sum_{n=0}^{\infty} \frac{L_{rn}}{2n+1} \left(\frac{4}{L_r}\right)^n \binom{2n}{n}^{-1} = \frac{L_r \pi}{2}.$$

Solution by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

From [2, Theorem 2.4], we know that

$$\sum_{n=0}^{\infty} \frac{4^n x^n}{(2n+1)\binom{2n}{n}} = \frac{1}{x}\sqrt{\frac{x}{1-x}}\, \arctan\left(\sqrt{\frac{x}{1-x}}\right), \qquad |x| < 1.$$

Setting $x = \alpha^r / L_r$, we immediately obtain (recall that r is even)

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\alpha^{rn}}{(2n+1)\binom{2n}{n}} \left(\frac{4}{L_r}\right)^n &= \frac{L_r}{\alpha^r}\sqrt{\frac{\alpha^r}{L_r-\alpha^r}}\, \arctan\left(\sqrt{\frac{\alpha^r}{L_r-\alpha^r}}\right) \\
&= \frac{L_r}{\alpha^r}\sqrt{\frac{\alpha^r}{\beta^r}}\, \arctan\left(\sqrt{\frac{\alpha^r}{\beta^r}}\right) \\
&= L_r \arctan\big(\alpha^r\big).
\end{aligned}$$

Similarly, with $x = \beta^r / L_r$, we obtain

$$\sum_{n=0}^{\infty} \frac{\beta^{rn}}{(2n+1)\binom{2n}{n}} \left(\frac{4}{L_r}\right)^n = L_r \arctan\big(\beta^r\big).$$

Adding the two identities yields

$$\sum_{n=0}^{\infty} \frac{L_{rn}}{(2n+1)\binom{2n}{n}} \left(\frac{4}{L_r}\right)^n = L_r \big[\arctan(\alpha^r) + \arctan(\beta^r)\big] = \frac{L_r \pi}{2},$$

where, in the last step, we have used the identity $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x)\,\frac{\pi}{2}$.

Editor's Note: Some solvers used the power series [1]

$$\sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x\, \arcsin x}{\sqrt{1-x^2}}$$

to derive their solutions.

References

[1] D. H. Lehmer, *Interesting series involving the central binomial coefficient*, Amer. Math. Monthly, **92** (1985), 452.

[2] R. Sprugnoli, *Sums of reciprocals of the central binomial coefficients*, Integers, **2006**, Article #A27.

Also solved by Ulrich Abel, Michel Bataille, Brian Bradie, Nandan Sai Dasireddy, Ángel Plaza, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, Seán M. Stewart, Andrés Ventas, and the proposer.