ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2024. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1331 Proposed by Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that $\dfrac{F_{n+2}^2}{F_{n-1}^2} + \dfrac{F_{n+2}}{F_n} + \dfrac{F_{n+2}}{F_{n+1}} \geq 9$ for all integers $n \geq 2$.

B-1332 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let T_n be the nth Tribonacci number defined by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$. Prove that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{T_n T_{n+1} T_{n+2} T_{n+4}} = -\frac{1}{16}$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{T_n T_{n+1} T_{n+4}} = \frac{3}{16}$.

B-1333 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Find all solutions of the equation

$$(F_a + 1)(F_b + 1)(F_c + 1) = 3F_aF_bF_c.$$

B-1334 Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.

For all integers $n \geq 3$, show that

$$\frac{(F_{n+1} - 1)^2}{(F_n - 1)(L_n - 1)} + \frac{(F_n - 1)^2}{(F_{n-1} - 1)(L_{n-1} - 1)} < F_{2n+1}.$$

B-1335 Proposed by Michel Bataille, Rouen, France.

Let the sequence $\{G_n\}_{n\geq 0}$ be defined by arbitrary $G_0, G_1 \in \mathbb{N}$, and the recurrence $G_{n+1} = G_n + G_{n-1}$ for any integer $n \geq 1$. If m and n are integers such that $m \geq 1$ and $n \geq 0$, prove that

$$\prod_{k=1}^{m} \frac{G_n + G_{n+2^{k+1}}}{G_{n+2^k}}$$

is a Fibonacci number.

SOLUTIONS

It's All About Catalan

B-1311 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 60.3, August 2022)

Let r be an integer. For any positive even integer n, prove that

$$\sum_{k=1}^{n} (-1)^k (F_{rk}F_{rk+r})^2 = \frac{(F_{rn}F_{rn+2r})^2}{L_{2r}}.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Let $n = 2m$. By summing up consecutive terms two by two on the left side, the identity can be written as

$$\sum_{k=1}^{m} \left[(F_{2rk}F_{2rk+r})^2 - (F_{2rk-r}F_{2rk})^2\right] = \frac{(F_{2rm}F_{2rm+2r})^2}{L_{2r}}.$$

Note that

$$(F_{2rk}F_{2rk+r})^2 - (F_{2rk-r}F_{2rk})^2 = F_{2rk}^2(F_{2rk+r}^2 - F_{2rk-r}^2) = F_{2rk}^2 F_{2r} F_{4rk},$$

according to Catalan's identity $F_n^2 - (-1)^{n-m}F_m^2 = F_{n-m}F_{n+m}$. Thus, the identity to be proved becomes

$$\sum_{k=1}^{m} F_{2rk}^2 F_{4rk} = \frac{(F_{2rm}F_{2rm+2r})^2}{F_{4r}},$$

which may be proved by induction. For $m = 1$, the identity holds. If it holds for some integer $m \geq 1$, then it is enough to prove that

$$\frac{(F_{2rm}F_{2rm+2r})^2}{F_{4r}} + F_{2r(m+1)}^2 F_{4r(m+1)} = \frac{(F_{2r(m+1)}F_{2r(m+1)+2r})^2}{F_{4r}},$$

or

$$\frac{F_{2rm}^2}{F_{4r}} + F_{4r(m+1)} = \frac{F_{2rm+4r}^2}{F_{4r}}.$$

The last identity follows from $F_{2rm+4r}^2 - F_{2rm}^2 = F_{4r}F_{4rm+4r}$, which is, again, a consequence of Catalan's identity.

Solution 2 by the Proposer.

As a special case of Catalan's identity, we have

$$F_{a+b}^2 - F_{a-b}^2 = F_{2a}F_{2b}.$$

Letting $n = 2m$, we have

$$\begin{aligned}
L_{2r}\sum_{k=1}^{n}(-1)^k(F_{rk}F_{r(k+1)})^2 &= L_{2r}\sum_{k=1}^{m}\left[-(F_{r(2k-1)}F_{2rk})^2 + (F_{2rk}F_{r(2k+1)})^2\right] \\
&= \sum_{k=1}^{m} L_{2r}F_{2rk}^2\big(F_{2rk+r}^2 - F_{2rk-r}^2\big) = \sum_{k=1}^{m} L_{2r}F_{2rk}^2 \cdot F_{4rk}F_{2r} \\
&= \sum_{k=1}^{m} F_{2rk}^2 \cdot F_{4rk}F_{4r} = \sum_{k=1}^{m} F_{2rk}^2\big(F_{2rk+2r}^2 - F_{2rk-2r}^2\big) \\
&= \sum_{k=1}^{m}\big(F_{2r(k+1)}^2F_{2rk}^2 - F_{2rk}^2F_{2r(k-1)}^2\big) = F_{2r(m+1)}^2F_{2rm}^2 - F_{2r}^2F_0^2 \\
&= (F_{rn}F_{rn+2r})^2.
\end{aligned}$$

Therefore, we obtain the desired identity.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Kristen Hartz (undergraduate), Won Kyun Jeong, Muzahim Mamedov, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, and Andrés Ventas.

Fibonacci and Lucas Subscripts

B-1312 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 60.3, August 2022)

For any positive integer n, find closed form expressions for the sums

$$\sum_{k=1}^{n} L_{F_k}L_{F_{k+1}}F_{F_k}F_{F_{k+1}} \qquad \text{and} \qquad \sum_{k=1}^{n} L_{L_k}L_{L_{k+1}}F_{L_k}F_{L_{k+1}}.$$

Solution 1 by Jason L. Smith, Richland Community College, Decatur, IL.

The summand in each sum has the form of $F_{G_k}L_{G_k}F_{G_{k+1}}L_{G_{k+1}}$ for a generalized Fibonacci sequence $\{G_k\}_{k=1}^{\infty}$. Using the Fibonacci double-angle identity $F_mL_m = F_{2m}$, this immediately becomes $F_{2G_k}F_{2G_{k+1}}$. Using the product formula $F_sF_t = \frac{1}{5}\left[L_{s+t} - (-1)^t L_{s-t}\right]$, the summand can be written as

$$\frac{1}{5}\left(L_{2G_{k+1}+2G_k} - L_{2G_{k+1}-2G_k}\right) = \frac{1}{5}\left(L_{2G_{k+2}} - L_{2G_{k-1}}\right).$$

We now see that each sum telescopes:

$$\begin{aligned}\sum_{k=1}^{n} F_{G_k}L_{G_k}F_{G_{k+1}}L_{G_{k+1}} &= \frac{1}{5}\sum_{k=1}^{n}\left(L_{2G_{k+2}} - L_{2G_{k-1}}\right)\\ &= \frac{1}{5}\left(L_{2G_{n+2}} + L_{2G_{n+1}} + L_{2G_n} - L_{2G_2} - L_{2G_1} - L_{2G_0}\right).\end{aligned}$$

The first sum uses $G_k = F_k$, in which case we also observe that $L_{2F_2} + L_{2F_1} + L_{2F_0} = L_2 + L_2 + L_0 = 8$. Therefore, the first sum becomes

$$\sum_{k=1}^{n} F_{F_k}L_{F_k}F_{F_{k+1}}L_{F_{k+1}} = \frac{1}{5}\left(L_{2F_{n+2}} + L_{2F_{n+1}} + L_{2F_n} - 8\right).$$

The second sum uses $G_k = L_k$, where we observe that $L_{2L_2} + L_{2L_1} + L_{2L_0} = L_6 + L_2 + L_4 = 28$. Thus, the second sum becomes

$$\sum_{k=1}^{n} F_{L_k}L_{L_k}F_{L_{k+1}}L_{L_{k+1}} = \frac{1}{5}\left(L_{2L_{n+2}} + L_{2L_{n+1}} + L_{2L_n} - 28\right).$$

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

Using the product formulas (which are easy to derive from Binet's formulas)

$$F_aL_b = F_{a+b} + (-1)^b F_{a-b}, \qquad \text{and} \qquad L_aF_b = F_{a+b} - (-1)^b F_{a-b},$$

we find

$$\begin{aligned}\sum_{k=1}^{n} L_{F_k}L_{F_{k+1}}F_{F_k}F_{F_{k+1}} &= \sum_{k=1}^{n}\left(F_{F_{k+1}}L_{F_k}\right)\left(L_{F_{k+1}}F_{F_k}\right)\\ &= \sum_{k=1}^{n}\left[F_{F_{k+2}} + (-1)^{F_k}F_{F_{k-1}}\right]\left[F_{F_{k+2}} - (-1)^{F_k}F_{F_{k-1}}\right] = \sum_{k=1}^{n}\left(F_{F_{k+2}}^2 - F_{F_{k-1}}^2\right)\\ &= F_{F_{n+2}}^2 + F_{F_{n+1}}^2 + F_{F_n}^2 - F_{F_0}^2 - F_{F_1}^2 - F_{F_2}^2 = F_{F_{n+2}}^2 + F_{F_{n+1}}^2 + F_{F_n}^2 - 2.\end{aligned}$$

Similarly,

$$\sum_{k=1}^{n} L_{L_k}L_{L_{k+1}}F_{L_k}F_{L_{k+1}} = \sum_{k=1}^{n}\left(F_{L_{k+2}}^2 - F_{L_{k-1}}^2\right) = F_{L_{n+2}}^2 + F_{L_{n+1}}^2 + F_{L_n}^2 - 6.$$

Editor's Notes: Greubel obtained $\frac{1}{5}\left(L_{F_n}^2 + L_{F_{n+1}}^2 + L_{F_{n+2}}^2 - 6\right)$, and $\frac{1}{5}\left(L_{L_n}^2 + L_{L_{n+1}}^2 + L_{L_{n+2}}^2 - 11\right)$.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Kenny B. Davenport, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Albert Stadler, Seán M. Stewart, David Terr, and the proposer.

A Problematic Problem

B-1313 Proposed by Daniel Văcaru, Economical College "Maria Teiuleanu," Piteşti, Romania, and Mihály Bencze, "Aprily Lajos," Braşov, Romania. (Vol. 60.3, August 2022)

For $a \leq -1$, show that

$$\sum_{k=1}^{n} F_k(F_{n+2} - F_k - 1)^a \geq \left(\frac{n-1}{n}\right)^a (F_{n+2} - 1)^{a+1},$$

$$\sum_{k=1}^{n} F_k^2(F_nF_{n+1} - F_k)^a \geq \left(\frac{n-1}{n}\right)^a (F_nF_{n+1})^{a+1}.$$

Editor's Notes: There is a typo in the second inequality: F_k should be F_k^2. The error was corrected in the November 2022 issue. The solution will appear in the next issue.

It's Catalan Again!

B-1314 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany. (Vol. 60.3, August 2022)

Show that

$$\sum_{k=1}^{n} \frac{F_{4k}^2 - 1}{F_{2k}^2 + 1} = F_{4n+2} - (3n+1).$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

By Catalan's identity, we find

$$F_{2k}^2 + 1 = F_{2k-1}F_{2k+1},$$

and

$$F_{4k}^2 - 1 = F_{4k-2}F_{4k+2} = F_{2k-1}L_{2k-1}F_{2k+1}L_{2k+1}.$$

Thus,

$$\sum_{k=1}^{n} \frac{F_{4k}^2 - 1}{F_{2k}^2 + 1} = \sum_{k=1}^{n} L_{2k-1}L_{2k+1} = \sum_{k=1}^{n}(L_{4k} - L_2) = \left(\sum_{k=1}^{n} L_{4k}\right) - 3n.$$

We will now show that

$$\sum_{k=1}^{n} L_{4k} = F_{4n+2} - 1$$

by induction. For $n = 1$ we have

$$\sum_{k=1}^{1} L_{4k} = L_4 = 7 = 8 - 1 = F_6 - 1 = F_{4\cdot1+2} - 1.$$

Now suppose

$$\sum_{k=1}^{n} L_{4k} = F_{4n+2} - 1$$

for some positive integer n. Then,

$$\sum_{k=1}^{n+1} L_{4k} = L_{4n+4} + F_{4n+2} - 1 = F_{4n+5} + F_{4n+3} + F_{4n+2} - 1 = F_{4n+6} - 1,$$

which completes the induction. Finally,

$$\sum_{k=1}^{n} \frac{F_{4k}^2 - 1}{F_{2k}^2 + 1} = F_{4n+2} - (3n+1).$$

Editor's Notes: By observing that $L_{4k} = F_{4k+1} + F_{4k-1} = F_{4k+2} - F_{4k-2}$, as Ohtsuka did, we can see that $\sum_{k=1}^{n} L_{4k} = F_{4n+2} - 1$ telescopes to $F_{4n+2} - 1$.

Also solved by Thomas Achammer, Michel Bataille, Brian B. Beasley, Charlee K. Cook, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, G. C. Greubel, Kristen Hartz (undergraduate), Won Kyun Jeong, Hiduyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, Seán M. Stewart, David Terr, Dan Weiner, and the proposer.

This Time, It Is d'Ocagne's Turn

B-1315 Proposed by Michel Bataille, Rouen, France.
(Vol. 60.3, August 2022)

For any positive integer n, let $U_n = \sum_{k=1}^{2n-1} \frac{1}{F_k F_{k+2}}$ and $V_n = \prod_{k=1}^{n} \frac{1}{2 - U_k}$. Prove that U_n is the ratio of two consecutive integers, and V_n is the ratio of two consecutive Fibonacci numbers. Lastly, evaluate $\prod_{n=1}^{\infty} \left(\sum_{k=1}^{2n} \frac{1}{F_k F_{k+2}} \right)$.

Solution by Albert Stadler, Herrliberg, Switzerland.

We have

$$\frac{1}{F_k F_{k+1}} - \frac{1}{F_{k+1} F_{k+2}} = \frac{F_{k+2} - F_k}{F_k F_{k+1} F_{k+2}} = \frac{1}{F_k F_{k+2}}$$

and

$$\begin{aligned} F_{2k} F_{2k+1} + 1 &= F_{2k-1} F_{2k+2}, \\ F_{2k+1} F_{2k+2} - 1 &= F_{2k} F_{2k+3}, \end{aligned}$$

by d'Ocagne's identity. Hence,

$$U_n = \sum_{k=1}^{2n-1} \frac{1}{F_k F_{k+2}} = \sum_{k=1}^{2n-1} \left(\frac{1}{F_k F_{k+1}} - \frac{1}{F_{k+1} F_{k+2}} \right) = 1 - \frac{1}{F_{2n} F_{2n+1}}$$

so that U_n is the ratio of the consecutive integers $F_{2n} F_{2n+1} - 1$ and $F_{2n} F_{2n+1}$. Furthermore,

$$\begin{aligned} V_n &= \prod_{k=1}^{n} \frac{1}{2 - U_k} = \prod_{k=1}^{n} \frac{1}{1 + \frac{1}{F_{2k} F_{2k+1}}} \\ &= \prod_{k=1}^{n} \frac{F_{2k} F_{2k+1}}{F_{2k} F_{2k+1} + 1} = \prod_{k=1}^{n} \frac{F_{2k} F_{2k+1}}{F_{2k-1} F_{2k+2}} = \frac{F_2 F_{2n+1}}{F_1 F_{2n+2}} = \frac{F_{2n+1}}{F_{2n+2}} \end{aligned}$$

is the ratio of two consecutive Fibonacci numbers. Finally,

$$\prod_{n=1}^{\infty}\left(\sum_{k=1}^{2n}\frac{1}{F_kF_{k+2}}\right)=\prod_{n=1}^{\infty}\left(1-\frac{1}{F_{2n+1}F_{2n+2}}\right)=\prod_{n=1}^{\infty}\left(\frac{F_{2n+1}F_{2n+2}-1}{F_{2n+1}F_{2n+2}}\right)$$
$$=\prod_{n=1}^{\infty}\frac{F_{2n}F_{2n+3}}{F_{2n+1}F_{2n+2}}=\lim_{N\to\infty}\prod_{n=1}^{N}\frac{F_{2n}F_{2n+3}}{F_{2n+1}F_{2n+2}}=\lim_{N\to\infty}\frac{F_2F_{2N+3}}{F_3F_{2N+2}}=\frac{\alpha}{2}.$$

Also solved by Thomas Achammer, Brian Bradie, Charles K. Cook and Michael R. Bacon (jointly), Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Kristen Hartz (undergraduate), Won Kyun Jeong, Hideyuki Ohtsuka, Ángel Plaza, Seán M. Stewart, David Terr, Andrés Ventas, and the proposer.

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