ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2011. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

<u>B-1071</u> Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove the following identities:

(1) $F_{n-1}^4 + 4F_n^4 + 4F_{n+1}^4 + F_{n+2}^4 = 6F_{2n+1}^2$,

(2) $F_{n-1}^6 + 8F_n^6 + 8F_{n+1}^6 + F_{n+2}^6 = 10F_{2n+1}^3$.

B-1072 **Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain**

Let n be a positive integer. For any real number, $\gamma > 1$, show that

$$\frac{1}{\gamma}\sum_{k=1}^{n}\left(F_k^{2\gamma}L_k^{2(1-\gamma)} + (\gamma-1)L_k^2\right) \geq F_nF_{n+1}.$$

B-1073 **Proposed by M. N. Deshpande, Nagpur, India**

Three integers (a, b, c) form a Diophantine Triple (DT) if and only if $ab+1$, $ac+1$, and $bc+1$ are perfect squares. It is known that $(F_{2n},\ F_{2n+2},\ F_{2n+4})$ is a DT for every integer n. If n is odd, prove that there exists an integer m such that $(m - F_{2n+4}, m - F_{2n+2}, m - F_{2n})$ is a DT. Also, if $n = 2k+1$ and the corresponding m is denoted by m_k, derive a recurrence relation involving m_k.

B-1074 **Proposed by Pantelimon George Popescu, Bucureşt, România and José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain**

Let $n \geq 3$ be a positive integer. Prove that

$$\frac{1}{\sqrt{1-\frac{1}{F_n^2}}} + \frac{1}{\sqrt{1-\frac{1}{L_n^2}}} > \frac{2}{\sqrt{1-\left(\frac{F_{n+1}}{F_{2n}}\right)^2}}.$$

B-1075 **Proposed by Paul S. Bruckman, Nanaimo, BC, Canada**

The Fibonacci polynomials $F_n(x)$ may be defined by the following expression:

$$F_{n+1}(x) = \sum_{k=0}^{[n/2]}\binom{n-k}{k}x^{n-2k} \text{ for } n = 0, 1, 2, \ldots.$$

Prove the "inverse" relation:

$$x^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{n+1-2k}(x) \text{ for } n = 0, 1, 2, \ldots.$$

SOLUTIONS

A Quartic Inequality

B-1051 Proposed by Charles K. Cook, Sumter, SC (Vol. 46/47.3, August 2008/2009)

For all positive integers n show that $F_n^4 + L_n^4 - 6F_{2n} + 5 > 0$.

Solution by Sergio Falcón and Ángel Plaza, ULPGC, Spain

Let $A(n) = F_n^4 + L_n^4 - 6F_{2n}$. Since $F_n = F_{n+1} - F_{n-1}$, $L_n = F_{n+1} + F_{n-1}$, and $F_{2n} = (F_{n+1} - F_{n-1})F_n$,

$$\begin{aligned} A(n) &= (F_{n+1} - F_{n-1})^4 + (F_{n+1} + F_{n-1})^4 - 6(F_{n+1}^2 - F_{n-1}^2) \\ &= 2F_{n+1}^4 + 12F_{n+1}^2F_{n-1}^2 + 2F_{n-1}^4 - 6F_{n+1}^2 + 6F_{n-1}^2 \\ &= 2F_{n+1}^2(F_{n+1}^2 - 3) + 2F_{n-1}^2(F_{n-1}^2 + 6F_{n+1}^2 + 3). \end{aligned}$$

Note that for $n = 1$, $A(1) + 5 = 2 \cdot 1(1 - 3) + 5 = 1 > 0$. For $n \geq 2$, since $F_{n+1} \geq 2$, then $A(n) > 1$, and therefore, $A(n) + 5 > 0$.

Also solved by Gurdial Arora and Andrea Edwards (jointly), Paul S. Bruckman, G. C. Greubel, Russell J. Hendel, Jaraslav Seibert, James A. Sellers, and the proposer.

A Convoluted Identity

B-1052 Proposed by Br. J. Mahon, Australia (Vol. 46/47.3, August 2008/2009)

Prove that

$$\sum_{r=2}^{\infty} \frac{F_r^2 + (-1)^r r^2}{F_{r+1}^{(1)} F_r^{(1)}} = \frac{5}{\alpha}$$

where $\{F_n^{(1)}\}$ is the sequence of first order convolutions of the Fibonacci numbers defined by

$$F_n^{(1)} = \sum_{i=0}^{n} F_{n-i} F_i.$$

Solution by Paul S. Bruckman, Surrey, BC, Canada

Without too much effort, we may show that $F_n^{(1)} = (1/5)\{nL_n - F_n\}$, $n = 0, 1, 2, \ldots$. For example, a convolution approach yields the desired formula. Note that $F_n^{(1)} > 0$ if $n \geq 2$. Consider the partial sum

$$S_n = \sum_{r=2}^{n} \{(F_r)^2 + (-1)^r r^2\}/F_{r+1}^{(1)} F_r^{(1)}, n \geq 2.$$

Next, we show that $S_n = \sum_{r=2}^{n}\{A_{r+1}/F_{r+1}^{(1)} - A_r/F_r^{(1)}\}$, where $A_r = (5/2)(r-1)F_r$. To verify this, note that $A_{r+1}F_r^{(1)} - A_rF_{r+1}^{(1)}$

$$\begin{aligned} &= (1/2)rF_{r+1}\{rL_r - F_r\} - (1/2)(r-1)F_r\{(r+1)L_{r+1} - F_{r+1}\} \\ &= r^2/2\,L_rF_{r+1} - r/2\,F_{r+1}F_r - (r^2-1)/2\,L_{r+1}F_r + (r-1)/2\,F_{r+1}F_r. \end{aligned}$$

Now note that $L_rF_{r+1} - L_{r+1}F_r = 2(-1)^r$, and $L_{r+1} = F_{r+1} + 2F_r$. Therefore,

$$A_{r+1}F_r^{(1)} - A_rF_{r+1}^{(1)} = r^2(-1)^r + (F_r)^2,$$

which proves the indicated telescoping formula for S_n.

Thus, we easily evaluate S_n as $S_n = A_{n+1}/F_{n+1}^{(1)} - A_2/F_2^{(1)}$. We note that

$$\begin{aligned} A_{n+1}/F_{n+1}^{(1)} &= (25/2)nF_{n+1}/\{(n+1)L_{n+1} - F_{n+1}\} \\ &= (25/2)nF_{n+1}/\{nL_{n+1} + 2F_n\} \sim (5\sqrt{5}/2)n\alpha^{n+1}/\{n\alpha^{n+1} + 2\alpha^n/\sqrt{5}\} \\ &= (5\sqrt{5}/2)\{1 + 0(1/n)\}^{-1} \\ &= (5\sqrt{5}/2)\{1 + 0(1/n)\}, \end{aligned}$$

as $n \to \infty$. Also, $A_2/F_2^{(1)} = 5/2$. Therefore, $S_n \to S$ as $n \to \infty$, where $S = (5/2)\{\sqrt{5} - 1\} = 5/\alpha$.

Also solved by the proposer.

Cubic Root Inequality

B-1053 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad Politècnica de Cataluña, Barcelona, Spain (Vol. 46/47.3, August 2008/2009)

Let n be a nonnegative integer. Prove that

$$\frac{1}{F_{n+2}}\left(\sqrt[3]{F_nF_{n+1}} + \sqrt[3]{F_{n+2}F_{n+3}}\right) < \sqrt{6}.$$

Solution by Charles K. Cook, Sumter, SC

The identity $F_n + F_{n+2} = L_{n+1}$ will be used as needed. Note first that the arithmetic, geometric mean inequality of 3 integers, $(1, a, b)$ yields

$$\sqrt[3]{F_nF_{n+1}} \le \frac{1 + F_n + F_{n+1}}{3} \quad \text{and} \quad \sqrt[3]{F_{n+2}F_{n+3}} \le \frac{1 + F_{n+2} + F_{n+3}}{3}.$$

Summing the righthand side yields

$$\frac{2 + F_n + F_{n+1} + F_{n+2} + F_{n+3}}{3} = \frac{2 + L_{n+1} + L_{n+2}}{3} = \frac{2 + L_{n+3}}{3}.$$

Thus,

$$\frac{\sqrt[3]{F_nF_{n+1}} + \sqrt[3]{F_{n2}F_{n+3}}}{F_{n+2}} \le \frac{2 + L_{n+3}}{3F_{n+2}}.$$

Next, it is immediate that $4 \leq 2F_n + F_{n+2}$. This implies $4+2F_{n+1} \leq 3F_{n+2}$. Thus, $4+2F_{n+3} \leq 5F_{n+2}$ and so $4 + 2F_{n+4} \leq 7F_{n+2}$. Therefore, $4 + 2F_{n+4} + 2F_{n+2} \leq 9F_{n+2}$. It follows that $4 + 2L_{n+3} \leq 9F_{n+2}$ and $\frac{2+L_{n+3}}{3F_{n+2}} \leq \frac{3}{2} = 1.5 < \sqrt{6}$. Hence, the desired inequality is satisfied.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Russell J. Hendel, Jaraslav Seibert, and the proposer.

A Converging Fibonacci Quotient

B-1054 Proposed by H.- J. Seiffert, Berlin, Germany (Vol. 46/47.3, August 2008/2009)

Show that the sequence $\{x_n\}_{n\geq 1}$ defined recursively by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{F_n x_n + F_{n+1}}{F_n x_n + F_{n-1}} \text{ for } n \geq 1,$$

converges and find the limit.

Solution by Paul S. Bruckman, Surrey, BC, Canada

For the moment, suppose that $x_n \to x$ as $n \to \infty$. We also know that $F_n \sim \alpha^n/\sqrt{5}$ as $n \to \infty$. Therefore, our supposition implies $x = (x + \alpha)/(x - \beta)$, which yields the quadratic equation $x^2 - \beta^2 x - \alpha = 0$. Clearly, we must have $x > 0$. Solving the quadratic and rejecting the negative root, we obtain $x = (1/2)\{\beta^2 + (5+\alpha)^{1/2}\} \approx 1.477259996$. Therefore, if the limit exists, it must be equal to this last value.

Consider an auxiliary sequence $\{y_n\}$ defined as $y_1 = 1$, $y_{n+1} = (y_n+\alpha)/(y_n-\beta)$, $n = 1, 2, \ldots$. We see that $y_n \to x$ as $n \to \infty$. Moreover, the convergence is faster than is the case with the original sequence $\{x_n\}$. Next, note that $y_{n+1} = 1 + 1/(y_n - \beta)$. Let $z_n = y_n - \beta$. Then $z_{n+1} = \alpha + 1/z_n$, with $z_1 = \alpha$. We see that z_n converges to some value, say z, as $n \to \infty$; moreover, $z = [\alpha, \alpha, \alpha, \ldots] = [\overline{\alpha}]$, an infinite periodic simple continued fraction. It follows that x exists, and that $x = z + \beta$. In addition, $z_n = [\overbrace{\alpha, \alpha, \ldots, \alpha}^{n \text{ terms}}]$, and $y_n = z_n + \beta$. Alternatively, we may say that $x = [1; \overline{\alpha}]$.

Also solved by G. C. Greubel, Russell J. Hendel, and the proposer.

Diaphontine Equation But Fibonacci Solutions

B-1055 Proposed by G. C. Greubel, Newport News, VA (Vol. 46/47.3, August 2008/2009)

Find all integer solutions to the equation

$$x^2 + 6xy + 4y^2 = 4.$$

Solution by Herman Roelants, Institute of Philosophy, University of Louvain, Belgium

Letting x (necessarily even) $= 2X$ leads to $4(X + y)^2 + 4Xy = 4$. Now using $4Xy = (X+y)^2 - (X-y)^2$ becomes $(X-y)^2 - 5(X+y)^2 = -4$. It is well-known [1, p. 30–32] that all solutions of $a^2 - 5b^2 = -4$ in positive integers are given by the pairs $(a_n, b_n) = (L_{2n+1}, F_{2n+1})$.

We now easily derive that $x_n = 2X_n = L_{2n+1} + F_{2n+1} = F_{2n} + F_{2n+1} + F_{2n+1} = 2F_{2n+2}$ and

$$y_n = \frac{F_{2n+1} - L_{2n+1}}{2} = -F_{2n}.$$

So all integer solutions of the proposed equation are given by the pairs $(x = 2F_{2n+2}, y = -F_{2n})$ and $(y = 2F_{2n+2},\ x = -F_{2n})$, given the symmetrical roles of x and y.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Chichester, Ellis Horwood Ltd., 1989.

Also solved by Paul S. Bruckman, Charles K. Cook, Russell J. Hendel, Jesus Pulido and LuCana Santos (students), Ángel Plaza and Sergio Falcón (jointly), Jaraslav Seibert, Paul Stockmeyer, and the proposer.

We wish to acknowledge Brian D. Beasly and George A. Heisert for solving problem B-1049, Ralph P. Grimaldi for solving problem B-1047, Russell J. Hendel for solving problems B-1048, B-1049, B-1050, and George A. Hisert for solving problem B-1047.