ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2013. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

<u>B-1111</u> Proposed by Mircea Merca, University of Craiova, Romania.

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \left\lceil \frac{F_k}{11} \right\rceil = \left\lceil \frac{F_{n+2}-1}{11} + \frac{3n}{5} \right\rceil.$$

B-1112 **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.**

Let n be a nonnegative integer. Show that

$$F_n^5 + F_{n+1}^5 + \frac{5}{7}\left(\frac{F_{n+2}^7 - F_{n+1}^7 - F_n^7}{F_{n+2}^2 - F_n F_{n+1}}\right)$$

is a fifth perfect power.

B-1113 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.**

Prove that

$$\frac{F_m^2}{(F_q F_n + F_{q+1} F_p)^2} + \frac{F_n^2}{(F_q F_p + F_{q+1} F_m)^2} + \frac{F_p^2}{(F_q F_m + F_{q+1} F_n)^2} \geq \frac{3}{F_{q+2}^2},$$

for any positive integers m, n, p, and q.

B-1114 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.**

Prove that

$$\sum_{k=1}^{n} \tan\left(1 + F_k^2\right)^2 > 4F_n F_{n+1}$$

for all positive integers n.

B-1115 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Given a positive integer m, prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} F_k^m\right)\left(\sum_{k=n+1}^{\infty}\frac{1}{F_k^m}\right) = \begin{cases} \frac{1}{L_m - 2} & \text{(if } m \text{ is even),} \\ \frac{1}{\sqrt{5}F_m - 2} & \text{(if } m \text{ is odd).} \end{cases}$$

SOLUTIONS

A Rate of Growth of Sum of Reciprocal Fibonacci Numbers

B-1091 **Proposed by Paul S. Bruckman, Nanaimo, BC, Canada.**
(Vol. 49.3, August 2011)

If

$$S_n = \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1},$$

show that $S_n \sim \frac{\alpha^{n-2}}{\sqrt{5}}$ as $n \to \infty$.

Solution by Kenneth B. Davenport, Dallas, PA, 18612.

The solution is an immediate consequence of a lemma in Ohtsukan and Nakamura's, *On the Sum of Reciprocal Fibonacci Numbers*, The Fibonacci Quarterly, Vol. 46/47, No. 2, May 2008/2009, pp. 153–159. More generally, the authors establish that

$$F_{n-2} < S_n < F_{n-2} + 1. \tag{1}$$

Since $F_n = \frac{\alpha^n}{\sqrt{5}}(1 + O(\alpha^{-2n}))$, where O denotes "big Oh," (1) implies the desired result.

Harris Kwong proved

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k^r} \right)^{-1} \sim \frac{\alpha^{(n-1)r}(\alpha^r - 1)}{5^{r/2}}$$

and G. C. Greubel proved

$$\left(\sum_{k=r}^{\infty} \frac{1}{F_{ak+b}} \right)^{-1} \sim \frac{\alpha^a - 1}{\sqrt{5}} \; \alpha^{an+b-a}.$$

Also solved by M. N. Deshpande, G. C. Greubel, Russell J. Hendell, Robinson Higuita, Ümit I. Şlak, Anastasios Kotronis, Harris Kwong, and the proposer.

Back Again!

B-1092 **Proposed by José Luis Díaz-Barrero, Technical University of Catalonia, Barcelona, Spain.**
(Vol. 49.3, August 2011)

Compute the sum

$$\sum_{n=0}^{\infty} \frac{1}{(5\alpha)^n (n+2)} \sum_{k=0}^{n} \frac{F_{k+1} F_{n-k+1}}{k+1}.$$

Solution. Kenneth Davenport and Zbigniew Jakubczyk pointed out that a solution was presented in The Fibonacci Quarterly, Vol. 43.2, May 2005, in the Advanced Problems section. The problem number is H-611 and the answer is

$$\frac{5\alpha^2}{2}\left[\ln\left(\frac{1+5\alpha^2}{4\alpha^2}\right)\right]^2.$$

Also solved by Paul Bruckman and the proposer.

<u>"Snake Oiled" Identity</u>

<u>B-1093</u> Proposed by Paul S. Bruckman, Nanaimo, BC, Canada (Vol. 49.3, August 2012)

Prove the following identity:

$$\sum_{k=0}^{[n/3]}(-1)^k\binom{n-2k}{2k}2^{n-3k}=F_{n+3}-1.$$

Solution I by Ümit I. Şlak, University of Southern California

We will show that the generating functions corresponding to the sequences on each side coincide. For the left-hand side we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}(-1)^k\binom{n-2k}{k}2^{n-3k}x^n &= \sum_{k=0}^{\infty}\sum_{n=3k}^{\infty}(-1)^k\binom{n-2k}{k}2^{n-3k}x^n\\ &= \sum_{k=0}^{\infty}(-1)^k x^{2k}2^{-k}\sum_{n=3k}^{\infty}\binom{n-2k}{k}2^{n-2k}x^{n-2k}\\ &= \sum_{k=0}^{\infty}(-1)^k x^{2k}2^{-k}\sum_{j=k}^{\infty}\binom{j}{k}(2x)j\\ &= \sum_{k=0}^{\infty}(-1)^k x^{2k}2^{-k}\sum_{j=0}^{\infty}\binom{j}{k}(2x)j.\end{aligned}$$

Now using the fact that $\sum_{j=0}^{\infty}\binom{j}{k}(2x)^j=\frac{(2x)^k}{(1-2x)^{k+1}}$ (see [1], p. 120), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}(-1)^k\binom{n-2k}{k}2^{n-3k}x^n=\frac{1}{1-2x}\sum_{k=0}^{\infty}\left(\frac{-x^3}{1-2x}\right)^k=\frac{1}{1-2x+x^3}.$$

Next we find the generating function of the right-hand side. Recalling that the generating function of the Fibonacci sequence is given by $\frac{x}{1-x-x^2}$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}(F_{n+3}-1)x^n=\frac{1}{x^3}\sum_{n=0}^{\infty}F_{n+3}x^{n+3}-\frac{1}{1-x} &= \frac{1}{x^3}\left(\frac{x}{1-x-x^2}-x-x^2\right)-\frac{1}{1-x}\\ &= \frac{1}{x^3-2x+1}.\end{aligned}$$

Since the generating functions coincide, we conclude that the given sequences are identical and we are done.

References

[1] Herbert S. Wilf, *Generating Functionology*, Academic Press, Inc., second ed., 1994.

Solution II by Annita Davis and Cecil Rousseau (jointly), University of Memphis, TN.

The binomial theorem and Cauchy integral formula combine to yield

$$\binom{n}{m} = \frac{1}{2\pi i}\int_C \frac{(z+1)^n}{z^{m+1}}\,dz,$$

where C is the circular contour $|z| = R$, and R is sufficiently large. Then

$$\begin{aligned}\sum_{k=0}^{[n/3]}(-1)^k\binom{n-2k}{k}2^{n-3k} &= 2^n\sum_{k=0}^{[n/3]}(-1)^k\left\{\frac{1}{2\pi i}\int_C\frac{(z+1)^{n-2k}}{z^{k+1}}\,dz\right\}2^{n-3k}\\ &= \frac{2^n}{2\pi i}\int_C\frac{(z+1)^n}{z}\sum_{k=0}^{[n/3]}(-\theta(z))^k\,dz\\ &= \frac{2^n}{2\pi i}\int_C\frac{(z+1)^n}{z}\frac{1+\theta(z)^{[n/3]+1}}{1+\theta(z)}\,dz\\ &= M_n + \Delta_n,\end{aligned}$$

where

$$M_n := \frac{2^n}{2\pi i}\int_C\frac{(z+1)^n}{z}\frac{1}{1+\theta(z)}\,dz,\ \ \Delta_n := \frac{2^n}{2\pi i}\int_C\frac{(z+1)^n}{z}\frac{\theta(z)^{[n/3]+1}}{1+\theta(z)}\,dz,$$

and $\theta(z) := (8z(z+1)^2)^{-1}$. The residue theorem will be used to calculate M_n. Note that the poles are $-\frac{1}{2}$, $\frac{-3+\sqrt{5}}{4}$, and $\frac{-3-\sqrt{5}}{4}$ and their respective residues are -1, $\frac{\alpha^{n+3}}{\sqrt{5}}$, and $\frac{\beta^{n+3}}{\sqrt{5}}$. Hence,

$$M_n = \frac{\alpha^{n+3}-\beta^{n+3}}{\sqrt{5}} - 1 = F_{n+3} - 1.$$

Now we prove $\Delta_n = 0$. In fact, since the integrand is a rational function $\frac{P}{Q}$ and $\deg Q - \deg P \geq 1 + 3^{[n/3]+1} - n \geq 2$, its integral over C is zero.

In summary,

$$\sum_{0}^{[n/3]}(-1)^k\binom{n-2k}{k} = M_n = F_{n+3} - 1.$$

Also solved by Kenneth B. Davenport, M. N. Deshpande, G. C. Greubel, Russell J. Hendel, Zbigniew Jakubczyk, Harris Kwong, Ángel Plaza and Sergio Falcón (jointly), and the proposer.

<u>The "Snake Oil Method" Again!</u>

<u>B-1094</u> Proposed by Paul S. Bruckman, Nanaimo, BC, Canada.
(Vol. 49.3, August 2011)

Prove the following identity:

$$\sum_{k=0}^{[n/3]} \binom{n-k}{2k} 2^{n-3k} = F_{2n} + 1.$$

Solution by Ángel Plaza and Sergio Falcón (jointly), Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain

We will use the "Snake Oil Method" [1]. The ordinary power series generating function (opsgf) of the left-hand side is

$$F(x) = \sum_{n\geq 0} x^n \sum_{k\leq /3} \binom{n-k}{2k} 2^{n-3k} = \sum_{k} 2^{-3k} \sum_{n\geq 3k} \binom{n-k}{2k} (2x)^n.$$

Using the identity [1, Equation (4.3.1), page 120]: $\sum_{r\geq 0} \binom{r}{j} x^r = \frac{x^j}{(1-x)^{j+1}} (j \geq 0)$, we have

$$\begin{aligned} F(x) &= \sum_{k} 2^{-2k} x^k \sum_{n\geq 3k} \binom{n-k}{2k} (2x)^{n-k} \\ &= \sum_{k} 2^{-2k} x^k \frac{(2x)^{2k}}{(1-2x)^{2k+1}} = \frac{1}{1-2x} \sum_{k} \frac{x^{3k}}{(1-2x)^{2k}} \\ &= \frac{1}{1-2x} \cdot \frac{1-2x}{1-4x+4x^2-x^3} = \frac{2x-1}{x^3-4x^2+4x-1}. \end{aligned}$$

For the right-hand side of the identity, by denoting $g_n = F_{2n}+1$, the sequence $\{g_n\}$ satisfies the recurrence relation $g_n = 3g_{n-1} - g_{n-2} - 1$, with initial values $g_0 = 1$ and $g_1 = 2$. So its opsgf, $G(x) = \sum_{n\geq 0} g_n x^n$, may be obtained as follows:

$$\begin{aligned} G(x) &= 3x \sum_{n\geq 1} g_{n-1} x^{n-1} - x^2 \sum_{n\geq 2} g_{n-2} x^{n-2} - \sum_{n\geq 0} x^n + 2 \\ &= 3xG(x) - x^2 G(x) - \frac{1}{1-x} + 2, \end{aligned}$$

and so $G(x) = \frac{2x-1}{x^3-4x^2+4x-1}$. Since $F(x) = G(x)$, the identity follows.

References

[1] Herbert S. Wilf, *Generating Functionology*, Academic Press, Inc., second ed., 1994.

Editor's Note: As noted by Annita Davis and Cecil Rousseau, their featured solution to Problem B-1093 works in almost exactly the same way here. It also works with slight variations for Problem H-704 which calls for the proof of the identity

$$\sum_{k=0}^{[n/4]} \binom{n-2k}{2k} 2^{n+1-4k} = P_{n+1} + n + 1,$$

where P_n is the nth Pell number.

Also solved by Kenneth B. Davenport, Annita Davis and Cecil Rousseau (jointly), M. N. Deshpande, Amos Gera, G. C. Greubel, Zbigniew Jakubczyk, Harris Kwong, and the proposer.

A Linear Combination of K-Fibonacci Numbers

B-1095 Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain. (Vol. 49.3, August 2011)

For any positive integer number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, with initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$. Let $\{a_{n,j}\}$ be the integer matrix defined by $a_{n,j} = \binom{n}{j} - k\binom{n}{j+1} - \binom{n}{j+2}$. Prove that

$$\sum_{j=0}^{n} a_{n,j}F_{k,j+1} = 1.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

From $a_{n,0} = 1 - \binom{n}{2} - nk$ we deduce the recurrent relation $a_{n+1,0} = a_{n,0} - (n + k)$. We also find, from Pascal's identity, that $a_{n,j} = a_{n-1,j} + a_{n-1,j-1}$ for $n, j \geq 1$. We will now use induction to prove that $\sum_{j=0}^{n} a_{n,j}F_{k,j+1} = 1$ *and* $\sum_{j=0}^{n} a_{n,j}F_{k,j} = n$ for all $n \geq 0$.

The assertion is obviously true when $n = 0$. Assume the identities hold for some integer $n \geq 0$. Then, because $F_{k,0} = a_{n,n+1} = 0$ and $F_{k,1} = 1$, we find

$$\begin{aligned}
\sum_{j=0}^{n+1} a_{n+1,j}F_{k,j} &= \sum_{j=1}^{n+1} a_{n,j} + a_{n,j-1}F_{k,j} \\
&= \sum_{j=0}^{n} a_{n,j}F_{k,j} + \sum_{j=0}^{n} a_{n,j}F_{k,j+1} \\
&= n + 1,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{n+1} a_{n+1,j}F_{k,j+1} &= a_{n+1,0} + \sum_{j=1}^{n+1} a_{n+1,j}F_{k,j+1} \\
&= a_{n,0} - (n+k) + \sum_{j=1}^{n+1}(a_{n,j} + a_{n,j-1})F_{k,j+1} \\
&= -(n+k) + \sum_{j=0}^{n} a_{n,j}F_{k,j+1} + \sum_{j=0}^{n} a_{n,j}F_{k,j+2} \\
&= 1 - (n+k) + \sum_{j=0}^{n} a_{n,j}(kF_{k,j+1} + F_{k,j}) \\
&= 1 - (n+k) + k\sum_{j=0}^{n} a_{n,j}F_{k,j+1} + \sum_{j=0}^{n} a_{n,j}F_{k,j} \\
&= 1 - (n+k) + k \cdot 1 + n \\
&= 1.
\end{aligned}$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, G. C. Greubel, Russell J. Hendel, Ümit I. Şlak, Zbigniew Jakubczyk, Harris Kwong, and the proposer.