ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2012. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1101 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

Prove that

$$\arctan\sqrt{\frac{F_n^2+F_{n+1}^2}{2}} + \arctan\sqrt{\frac{L_n^2+L_{n+1}^2}{2}} \geq \arctan\frac{F_{n+2}}{2} + \arctan\frac{L_{n+2}}{2}.$$

B-1102 Proposed by Diana Alexandrescu, University of Bucharest, Bucharest, Romania and José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain

Let n be a positive integer. Prove that

$$\left(\frac{\sqrt[3]{F_n^2}+\sqrt[3]{F_{n+1}^2}}{\sqrt[3]{F_{n+2}^2}}\right)\left(\frac{\sqrt[3]{L_n^2}+\sqrt[3]{L_{n+1}^2}}{\sqrt[3]{L_{n+2}^2}}\right) < \sqrt[3]{4}.$$

B-1103 Proposed by Hideyuki Ohtsuka, Saitama, Japan

If $a+b+c=0$ and $abc \neq 0$, find the value of

$$\frac{L_aL_bL_c}{F_aF_bF_c}\left(\frac{F_a}{L_a}+\frac{F_b}{L_b}+\frac{F_c}{L_c}\right).$$

B-1104 Proposed by Javier Sebastián Cortés (student), Universidad Distrital Francisco José de Caldas, Bogotá, Colombia

Prove that

$$F_{n+2k(k+1)}\sum_{i=0}^{2k} L_{n+2(k+1)i} = L_{n+2k(k+1)}\sum_{i=0}^{2k} F_{n+2(k+1)i}.$$

B-1105 Proposed by Paul S. Bruckman, Nanaimo, BC, Canada

Let

$$G_m(x)=\sum_{k=0}^{m+1}(-1)^{k(k+1)/2}\begin{bmatrix} m+1 \\ k\end{bmatrix}_F x^{m+1-k}, m=0,1,2,\ldots,$$

where $\left[\begin{smallmatrix} m+1 \\ k\end{smallmatrix}\right]_F$ is the *Fibonomial coefficient* $\frac{F_1F_2F_3\ldots F_{m+1}}{(F_1F_2F_3\ldots F_k)(F_1F_2F_3\ldots F_{m+1-k})}$, $1 \leq k \leq m$; also define

$$\begin{bmatrix} m+1 \\ 0\end{bmatrix}_F = \begin{bmatrix} m+1 \\ m+1\end{bmatrix}_F = 1.$$

Let $G_m(1)=U_m$. Prove the following, for $n=0,1,2,\ldots$:

(a) $U_{4n}=0$;
(b) $U_{4n+2}=2(-1)^{n+1}\{L_1L_2L_3\ldots L_{2n+1}\}^2$;
(c) $U_{2n+1}=(-1)^{(n+1)(n+2)/2}\{L_1L_3L_5\ldots L_{2n+1}\}$.

SOLUTIONS

From Cassini's Identity

B-1081 Proposed by Br. J. M. Mahon, Kensington, Australia (Vol. 49.1, February 2011)

Prove that

$$F_{n+1}^2F_{n-1}^2 - F_n^4 = \frac{(-1)^n}{5}\left[2L_{2n} + (-1)^n\right].$$

Solution by Carlos Rico, Student at Universidad Distrital Francisco José de Caldas (ITENU), Bogotá, Colombia.

Proof. Using Cassini's formula $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, [1, p. 74], it is easy to see that

$$\begin{aligned} F_{n+1}^2F_{n-1}^2 - F_n^4 &= (F_{n+1}F_{n-1})^2 - F_n^4 \\ &= (F_n^2 + (-1)^n)^2 - F_n^4 \\ &= (-1)^n[2F_n^2 + (-1)^n] \\ &= \frac{(-1)^n}{5}[10F_n^2 + 5(-1)^n]. \end{aligned}$$

Since $L_{2n} = 5F_n^2 + 2(-1)^n$, [1, p. 97],

$$\frac{(-1)^n}{5}[10F_n^2 + 5(-1)^n] = \frac{(-1)^n}{5}[2L_{2n} + (-1)^n],$$

and the desired equality follows. □

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by Shanon Michaels Arford, Gurdial Arora and Sindhu Unnithan (jointly), Brian D. Beasly, Scott H. Brown, Paul S. Bruckman, Jallisa Clifford, Kristopher Liggins and Dickson Toroitich (jointly, students), Charles K. Cook, Kenneth B. Davenport, Steve Edwards, Russell J. Hendel, Robinson Higuita and Sergio Mayorga (jointly, students), Zbigniew Jakubczyk, Harris Kwang, John Morrison, Jean Peterson, Ángel Plaza, Cecil Rousseau, Jaroslav Seibert, and the proposer.

An "Odd" Equality

B-1082 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain (Vol. 49.1, February 2011)

The k-Fibonacci numbers $F_n = F_{k,n}$ satisfy $F_{k,n+2} = kF_{k,n+1} + F_{k,n}, F_{k,0} = 0$, $F_{k,1} = 1$. Let $m > 1$ be a fixed integer. Prove that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1} F_{2mr+m}}{F_{mr}^2 F_{mr+m}^2} = \frac{1}{F_m^3}.$$

Solution by Russell J. Hendel, Towson University, Towson, MD.

The problem identity is, as stated, true for odd m but not for even m.

The $F_{k,n}$ are the generalized Lucas sequences [2], with Binet form, $\frac{u^n - v^n}{u-v}$, with u and v solutions of the equation $x^2 - kx - 1 = 0$, or alternatively, $u = \frac{k+\sqrt{D}}{2}$, $v = \frac{k-\sqrt{D}}{2}$, $D = k^2 + 4$. By a routine expansion of the Binet forms and straightforward simplification, we obtain for any positive integer m,

$$F_m F_{2mr+m} = (-1)^{m+1} F_{mr}^2 + F_{mr+r}^2,$$

which reduces for small values of m to known identities (e.g. [1, I11]).

When m is odd, substituting this last identity into the problem identity reduces it to the telescoping sum

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{(-1)^{r-1} F_{2mr+m}}{F_{mr}^2 F_{mr+m}^2} &= \lim_{n \to \infty} \frac{1}{F_m} \sum_{r=1}^{n} (-1)^{r-1} \left(\frac{1}{F_{mr}^2} + \frac{1}{F_{mr+m}^2} \right) \\
&= \lim_{n \to \infty} \frac{1}{F_m} \left(\frac{1}{F_m^2} - (-1)^n \frac{1}{F_{(n+1)m}^2} \right) \\
&= \frac{1}{F_m^3}.
\end{aligned}$$

Next, it suffices to disprove the problem identity for one even value of m say $m = 2$. But the infinite sum is an alternating sum of strictly positive values going to 0 and hence, the limit exists and is sandwiched between the partial sums,

$$P(n) = \sum_{r=1}^{n} \frac{(-1)^{r-1} F_{2mr+m}}{F_{mr}^2 F_{mr+m}^2}.$$

A straightforward computation shows that $P(n) : n = 1, 2, 3, 4, 5$ equals 0.1215278, 0.1181576, 0.1182566, 0.1182537, 0.1182538, and hence, $0.1182537 < P(\infty) < 0.1182538$, contradicting the problem assertion that the limit is $\frac{1}{F_m^3} = \frac{1}{2^3} = 0.125$.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, John Wiley, New York, 1989.

[2] `http://mathworld.wolfram.com/LucasSequence.html`.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Zbigniew Jakubczyk, and the proposer.

Much Ado About F_n

B-1083 Proposed by Br. J. M. Mahon, Kensington, Australia (Vol. 49.1, February 2011)

Find a closed form for

$$\sum_{j=0}^{[\frac{n}{2}]}(-1)^j \frac{n}{n-j}\binom{n-j}{j}F_{2n-3j}.$$

Solution by Cecil Rousseau, University of Memphis, Memphis, TN.

The sum equals F_n. A proof thereof is based on the identity

$$K(n,z) := \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j \frac{n}{n-j}\binom{n-j}{j}z^j = \left(\frac{1+\sqrt{1-4z}}{2}\right)^2 + \left(\frac{1-\sqrt{1-4z}}{2}\right)^n, \qquad (*)$$

which is in [1, p. 204]. By Binet's formula, the sum to be evaluated is

$$\begin{aligned} S_n &:= \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j \frac{n}{n-j}\binom{n-j}{j}\left(\frac{\alpha^{2n-3j}-\beta^{2n-3j}}{\sqrt{5}}\right) \\ &= \frac{\alpha^{2n}K(n,\alpha^{-3})-\beta^{2n}K(n,\beta^{-3})}{\sqrt{5}} \\ &= \frac{\alpha^{2n}K(n,-\beta^{3})-\beta^{2n}K(n,-\alpha^{3})}{\sqrt{5}}. \end{aligned}$$

Note that $1+4\alpha^3 = 1+4\alpha(1+\alpha) = (1+2\alpha)^2$, so

$$\frac{1+\sqrt{1+4\alpha^3}}{2} = 1+\alpha = \alpha^2 \quad \text{and} \quad \frac{1-\sqrt{1+4\alpha^3}}{2} = -\alpha.$$

The same relations hold with α replaced by β. Hence, (*) yields

$$\alpha^{2n}K(n,-\beta^3) = \alpha^{2n}\{(\beta^2)^n + (-\beta)^n\} = 1+\alpha^n$$

and $\beta^{2n}K(n,-\alpha^3) = 1+\beta^n$. Finally,

$$S_n = \frac{\alpha^n-\beta^n}{\sqrt{5}} = F_n.$$

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik *Concrete Mathematics*, second edition, Addison-Wesley, Reading, MA, 1998.

Zbigniew Jakobczyk notes that a similar formula holds for Lucas Numbers, precisely,

$$\sum_{j=0}^{[\frac{n}{2}]}(-1)^j \frac{n}{n-j}\binom{n-j}{j}L_{2n-3j} = L_n + 2.$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Zbigniew Jakubczyk, Ángel Plaza and Sergio Falcón (jointly), Jaroslav Seibert, and the proposer.

And by the AM-GM Inequality ....

B-1084 Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 49.1, February 2011)

Let n be a positive integer. Prove that

$$\left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{1+F_k^2}}\right)\left(\prod_{k=1}^{n}(1+F_k^2)\right)^{1/2n} \leq F_nF_{n+1}.$$

Solution by Ángel Plaza and Sergio Falcón (jointly), Universidad de Las Palmas de Gran Canaria, Spain.

Since $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$, the proposed inequality is equivalent to

$$\left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{1+F_k^2}}\right)\left(\prod_{k=1}^{n}\sqrt{1+F_k^2}\right)^{1/n} \leq \sum_{k=1}^{n} F_k^2.$$

This may be proved by the AM-GM and Chebyshev inequalities as follows:

$$\begin{aligned}\left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{1+F_k^2}}\right)\left(\prod_{k=1}^{n}\sqrt{1+F_k^2}\right)^{1/n} &\leq \left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{1+F_k^2}}\right)\left(\frac{\sum_{k=1}^{n}\sqrt{1+F_k^2}}{n}\right)\\ &\leq \sum_{k=1}^{n} F_k^2.\end{aligned}$$

The last inequality comes from the Chebyshev inequality since both sequences $\left\{\frac{F_k^2}{\sqrt{1+F_k^2}}\right\}$ and $\left\{\sqrt{1+F_k^2}\right\}$ have the same monotonicity. □

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Russell Jay Hendel, Robinson Higuita and Sergio Mayorga (jointly, students), Zbigniew Jakobczyk, Harris Kwang, and the proposer.

A Fibonacci Fraction

B-1085 Proposed by Carsten Elsner and Martin Stein, University of Applied Sciences, Hannover, Germany (Vol. 49.1, February 2011)

Let $(a_n)_{n\geq 1}$ be a sequence of positive integers. Let $q_0 = 1$, $q_1 = a_1$ and $q_n = a_nq_{n-1} + q_{n-2}$ for $n \geq 2$. Prove that

$$\frac{q_0+q_1+\cdots+q_{m-1}}{q_m} \leq \frac{F_{m+2}-1}{F_{m+1}}$$

for $m \geq 1$.

Solution by Paul S. Bruckman, Nanaimo, BC V9R 0B8, Canada.

From $q_m = a_m q_{m-1} + q_{m-2}$, $m \geq 2$, along with $\frac{q_1}{q_0} = a_1$, we see that $\frac{q_n}{q_{m-1}} = a_m + 1/(q_{m-1}/q_{m-2})$, which implies that $\frac{q_n}{q_{m-1}} = [a_m, a_{m-1}, \ldots, a_1]$; this is an abbreviation for the simple continued fraction $a_m + 1/a_{m-1} + \cdots + 1/a_1$. Note that $a_m < \frac{q_n}{q_{m-1}} < 1 + a_m$. Also, define r_m as follows:

$$r_m = \frac{\sum_{k=0}^{m-1} q_k}{q_m}.$$

Then

$$\begin{aligned} r_m &= \frac{q_0}{q_1} \cdot \frac{q_1}{q_2} \cdot \cdots \cdot \frac{q_{m-1}}{q_m} + \frac{q_1}{q_2} \cdot \frac{q_2}{q_3} \cdot \cdots \cdot \frac{q_{m-1}}{q_m} + \cdots + \frac{q_{m-2}}{q_{m-1}} \cdot \frac{q_{m-1}}{q_m} + \frac{q_{m-1}}{q_m} \\ &= \frac{1}{[a_m, \ldots, a_1]} \left\{ 1 + \frac{1}{[a_{m-1}, \ldots, a_1]} \left(1 + \frac{1}{[a_{m-2}, \ldots, a_1]} \{1 + \cdots\} \right) \right\}. \end{aligned}$$

From the last expression, we see that r_m (as a function of the a_k's) is maximized when all the a_k's are equal to one. It is easily verified, however, that in that case, $q_m = F_{m+1}$, hence,

$$r_m = \frac{\sum_{k=0}^{m-1} F_{k+1}}{F_{m+1}} = \frac{F_{m+2} - 1}{F_{m+1}}.$$

Therefore, $r_m \leq \frac{F_{m+2}-1}{F_{m+1}}$.

Also solved by Russell J. Hendel, Ángel Plaza and Sergio Falcón (jointly), and the proposer.

We wish to acknowledge Kenneth B. Davenport for solving problem B-1078.