ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2018. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1246 Proposed by Raphael Schumacher (student), ETH Zurich, Switzerland.

Prove that, for all integers $n \ge 0$,

$$\frac{F_{n-1}}{\sqrt{n+2}+\sqrt{n+1}} + \sum_{k=0}^{n} \frac{F_k}{\sqrt{k+1}+\sqrt{k}} = 2\sum_{k=1}^{n} \frac{F_{k-1}}{\sqrt{k+2}+\sqrt{k}} + \sqrt{2} - 1,$$

and deduce that

$$\frac{F_{n+1}}{\sqrt{n+2}+\sqrt{n+1}} = \sum_{k=0}^{n} \left(\frac{2}{\sqrt{k+3}+\sqrt{k+1}} - \frac{1}{\sqrt{k+1}+\sqrt{k}} \right) F_k - \frac{F_n}{\sqrt{n+3}+\sqrt{n+2}} + \sqrt{2} - 1.$$

B-1247 Proposed by Kenny B. Davenport, Dallas, PA.

Prove that, for all positive integers n,

$$\sum_{k=1}^{n} L_k^3 L_{k+1}^3 = \left(\sum_{k=1}^{n} L_k^2 L_{k+1} \right)^2 + 6 \sum_{k=1}^{n} L_k^2 L_{k+1}.$$

B-1248 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

For all positive integers n and a, prove that

$$\sum_{k=0}^{n} L_k(L_{k+1}^a + L_{k+2}^a) \leq (L_{n+2} - 1)(L_{n+2}^a + 1).$$

B-1249 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For positive integers s and t, prove that

$$\sum_{n=s}^{\infty} \frac{(-1)^{tn}}{\alpha^{(2t-1)n} F_n} = \sum_{n=t}^{\infty} \frac{(-1)^{sn}}{\alpha^{(2s-1)n} F_n}.$$

B-1250 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

Evaluate

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{F_{k+1}}{F_k F_{k+2} + 1} \tan^{-1} \frac{1}{F_{k+1}}.$$

Catalan Identity for the k-Fibonacci Numbers

B-1226 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 56.2, May 2018)

For any positive integer k, the k-Fibonacci and k-Lucas sequences, denoted $\{F_{k.n}\}_{n\geq 0}$ and $\{L_{k,n}\}_{n\geq 0}$ respectively, are both defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. Prove that, for any positive integer n,

$$\sum_{j=1}^{n} F_{k,j}^2 F_{k,2j} = \frac{F_{k,n}^2 F_{k,n+1}^2}{k}.$$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

For brevity, let $u_n = F_{k,n}$ and $v_n = L_{k,n}$. Use the Binet forms $u_n = \frac{p^n - q^n}{p-q}$ and $v_n = p^n + q^n$, where $p = \frac{k+\sqrt{k^2+4}}{2}$ and $q = \frac{k-\sqrt{k^2+4}}{2}$. Observe that $p + q = k$, $pq = -1$, and $p - q = \sqrt{k^2 + 4}$. It is not difficult to see that $u_{2n} = u_n v_n$ from these relations. It can also be shown that $p^2 + 1 = \sqrt{k^2 + 4} \cdot p$, and $q^2 + 1 = -\sqrt{k^2 + 4} \cdot q$. From these, we find that $u_n + u_{n+2} = v_{n+1}$.

Now, we can prove our sum using induction. First observe that $\sum_{j=1}^{1} u_j^2 u_{2j} = u_1^2 u_2 = k$, and $\frac{u_1^2 u_2^2}{k} = k$. Assume the identity holds for some integer $n \geq 1$. Next, consider the sum with upper limit $n + 1$:

$$\sum_{j=1}^{n+1} u_j^2 u_{2j} = \frac{u_n^2 u_{n+1}^2}{k} + u_{n+1}^2 u_{2n+2} = \frac{u_{n+1}^2}{k} \left(u_n^2 + k u_{n+1} v_{n+1}\right).$$

Next, square both sides of the recursion $u_{n+2} = ku_{n+1} + u_n$ to obtain

$$\begin{aligned} u_{n+2}^2 &= u_n^2 + ku_{n+1}\left(ku_{n+1} + 2u_n\right) \\ &= u_n^2 + ku_{n+1}\left(u_{n+2} + u_n\right) \\ &= u_n^2 + ku_{n+1}v_{n+1}. \end{aligned}$$

This implies $\sum_{j=1}^{n+1} u_j^2 u_{2j} = \frac{u_{n+1}^2 u_{n+2}^2}{k}$, which proves the result.

Editor's Note: Fedak's solution is essentially the same as the one presented above, except that he used the identities $ku_{n+1} = u_{n+2} - u_n$ and $v_{n+1} = u_{n+2} + u_n$ to conclude that

$$ku_{n+1}v_{n+1} = (u_{n+2} - u_n)(u_{n+2} + u_n) = u_{n+2}^2 - u_n^2.$$

Several solvers remarked that this identity is a special case of the Catalan's Identity for the k-Fibonacci numbers, see [1, Proposition 7] or [2, Proposition 3].

References

[1] S. Falcón and Á. Plaza, On the Fibonacci k-numbers, *Chaos, Solitons & Fractals*, **32** (2007), 1615–1624.
[2] S. Falcón and Á. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, *Chaos, Solitons & Fractals*, **33** (2007), 38–49.

Also solved by Alexandru Atim, Brian D. Beasley, Kenny B. Davenport, Ivan V. Fedak, Dmitry Fleischman, Robert Frontczak, Wei-Kai Lai, Santiago Alzate Suárez and Kevin Darío López Rodríguez (students) (jointly), David Terr, and the proposer.

Generating Function of the Catalan Numbers

B-1227 Proposed by Kenny B. Davenport, Dallas, PA.
(Vol. 56.2, May 2018)

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the nth Catalan number. Find the closed form expressions for the sums

$$\sum_{n=0}^{\infty} \frac{C_n F_n}{8^n}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{C_n L_n}{8^n}.$$

Solution by Lauren G. Mcanany (student), California University of Pennsylvania, California, PA.

We will find a closed expression for a more general sum; that is, we will prove that

$$\sum_{n=0}^{\infty} \frac{C_n G_n}{8^n} = 4\left(2a - b + \frac{4b - 7a}{\sqrt{10}}\right), \tag{1}$$

where $\{G_n\}_{n\geq 0}$ is the generalized Fibonacci sequence with $G_1 = a$, $G_2 = b$, and $G_n = G_{n-1} + G_{n-2}$, for $n \geq 3$,, and $a, b \in \mathbb{Z}$.

The generating function of the Catalan numbers is known to be

$$\mathcal{C}(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}, \qquad \text{for } |x| < \frac{1}{4}; \tag{2}$$

see [1, p. 345]. We also use the identity [2, p. 111]

$$G_n = \frac{c\alpha^n - d\beta^n}{\sqrt{5}}, \quad \text{for all } n \in \mathbb{Z},$$

where $c = a + (a-b)\beta$ and $d = a + (a-b)\alpha$, to obtain

$$\sum_{n=0}^{\infty} \frac{C_n G_n}{8^n} = \frac{c\,\mathcal{C}\big(\frac{\alpha}{8}\big) - d\,\mathcal{C}\big(\frac{\beta}{8}\big)}{\sqrt{5}}.$$

Since $\frac{\alpha}{8}, \frac{\beta}{8} \in \big(-\frac{1}{4}, \frac{1}{4}\big)$, we can apply (2) to evaluate the infinite sum:

$$\sum_{n=0}^{\infty} \frac{C_n G_n}{8^n} = \frac{4c}{\sqrt{5}\,\alpha}\left(1 - \sqrt{\frac{2-\alpha}{2}}\right) - \frac{4d}{\sqrt{5}\,\beta}\left(1 - \sqrt{\frac{2-\beta}{2}}\right).$$

Since $2 - \beta = \alpha^2$, we find $\sqrt{2-\beta} = \alpha$. Similarly, $2 - \alpha = \beta^2$ yields $\sqrt{2-\alpha} = -\beta$ (because $\beta < 0$). Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{C_n G_n}{8^n} &= \frac{4}{\sqrt{5}}\left[\left(\frac{c}{\alpha} - \frac{d}{\beta}\right) + \frac{1}{\sqrt{2}}\left(\frac{c\beta}{\alpha} + \frac{d\alpha}{\beta}\right)\right] \\ &= \frac{4}{\sqrt{5}}\left[\sqrt{5}\,(2a-b) + \frac{1}{\sqrt{2}}\,(4b-7a)\right] \\ &= 4\left(2a - b + \frac{4b-7a}{\sqrt{10}}\right).\end{aligned}$$

This proves (1). If we take $a = b = 1$ in (1), then we can see that

$$\sum_{n=0}^{\infty} \frac{C_n F_n}{8^n} = 4\left(1 - \frac{3}{\sqrt{10}}\right) = 4 - \frac{6}{5}\sqrt{10}.$$

If we now take $a = 1$ and $b = 3$ in (1), then

$$\sum_{n=0}^{\infty} \frac{C_n L_n}{8^n} = 4\left(-1 + \frac{5}{\sqrt{10}}\right) = -4 + 2\sqrt{10}.$$

References

[1] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, Oxford, 2009.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by Eric Blom, Khristo N. Boyadzhiev, Ivan V. Fedak, Dmitry Fleischman, Robert Frontczak, Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, Albert Stadler, Santiago Alzate Suárez (student), David Terr, Dan Weiner, and the proposer.

A Double Sum of Triple Products

B-1228 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 56.2, May 2018)

For any integer $n \geq 0$, find the closed form expressions for the sums

(i) $S_n = \displaystyle\sum_{i=0}^{n}\sum_{j=0}^{n} L_{3^i} L_{3^j} L_{2(3^i - 3^j)}$;

(ii) $T_n = \sum_{i=0}^{n}\sum_{j=0}^{n} F_{2\cdot 5^i} F_{2\cdot 5^j} L_{3(5^i-5^j)}$.

Solution by Ehren Metcalfe, Barrie, Ontario, Canada.

(i) Expand the body of S_n using the product formula $L_aL_b = (-1)^b L_{a-b} + L_{a+b}$ [1, Identity 17a] to show that the inner sum telescopes:

$$\begin{aligned} S_n &= \sum_{i=0}^{n}\sum_{j=0}^{n} L_{3^i} L_{3^j} L_{2(3^i-3^j)} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} \left(L_{3^{j+1}-2\cdot 3^i} + L_{2\cdot 3^i - 3^j}\right) L_{3^i} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} \left(L_{3^{j+1}-3^i} - L_{3^j-3^i} - L_{3^{i+1}-3^{j+1}} + L_{3^{i+1}-3^j}\right) \\ &= \sum_{i=0}^{n} \left(L_{3^{n+1}-3^i} - L_{1-3^i} - L_{3^{i+1}-3^{n+1}} + L_{3^{i+1}-1}\right). \end{aligned}$$

It follows from $L_{-a} = (-1)^a L_a$ that $L_{p-q} = L_{q-p}$ for odd integers p and q. We use this to make substitutions in S_n to see that the remaining sum telescopes:

$$S_n = \sum_{i=0}^{n} \left(L_{3^{n+1}-3^i} - L_{3^{n+1}-3^{i+1}} + L_{3^{i+1}-1} - L_{3^i-1}\right) = 2L_{3^{n+1}-1} - 4.$$

(ii) Expand T_n by applying the product formula $F_aF_b = \frac{1}{5}\left(L_{a+b} - (-1)^b L_{a-b}\right)$ [1, Identity 17b], followed by two applications of Identity 17a, to show that the inner sum telescopes:

$$\begin{aligned} T_n &= \sum_{i=0}^{n}\sum_{j=0}^{n} F_{2\cdot 5^i} F_{2\cdot 5^j} L_{3(5^i-5^j)} \\ &= \frac{1}{5}\sum_{i=0}^{n}\sum_{j=0}^{n} \left(L_{2(5^i+5^j)} - L_{2(5^i-5^j)}\right) L_{3(5^i-5^j)} \\ &= \frac{1}{5}\sum_{i=0}^{n}\sum_{j=0}^{n} \left(L_{5^{j+1}-5^i} - L_{5^j-5^i} - L_{5^{i+1}-5^{j+1}} + L_{5^{i+1}-5^j}\right) \\ &= \frac{1}{5}\sum_{i=0}^{n} \left(L_{5^{n+1}-5^i} - L_{1-5^i} - L_{5^{i+1}-5^{n+1}} + L_{5^{i+1}-1}\right). \end{aligned}$$

Therefore,

$$T_n = \frac{1}{5}\sum_{i=0}^{n} \left(L_{5^{n+1}-5^i} - L_{5^{n+1}-5^{i+1}} + L_{5^{i+1}-1} - L_{5^i-1}\right) = \frac{1}{5}\left(2L_{5^{n+1}-1} - 4\right).$$

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover, 2008.

Editor's Note: Davenport used the identities

$$\begin{aligned} L_xL_yL_z &= L_{x+y+z} + (-1)^x L_{-x+y+z} + (-1)^y L_{x-y+z} + (-1)^z L_{x+y-z}, \\ F_xF_yL_z &= L_{x+y+z} - (-1)^x L_{-x+y+z} - (-1)^y L_{x-y+z} + (-1)^z L_{x+y-z}, \end{aligned}$$

to derive the same telescopic sums. He refers the interested readers to the solution to Elementary Problem B-1203 that appeared in Volume 56.1 (2018), pages 85–86.

Also solved by Kenny B. Davenport, Ivan V. Fedak, and the proposer.

Stirling Approximation of Double Factorial

B-1229 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Bazău, Romania.
(Vol. 56.2, May 2018)

Let $m, p \geq 0$. Evaluate

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{\big((2n+1)!!\big)^{m+1} F_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{\big((2n-1)!!\big)^{m+1} F_n^{p(m+1)}}}{n^m} \right),$$

and

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{\big((2n+1)!!\big)^{m+1} L_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{\big((2n-1)!!\big)^{m+1} L_n^{p(m+1)}}}{n^m} \right).$$

Solution by David Terr, Oceanside, CA.

We define functions $\mathcal{F}(n; m, p)$ and $\mathcal{L}(n; m, p)$ as

$$\begin{aligned} \mathcal{F}(n; m, p) &= \frac{\sqrt[n]{\big((2n-1)!!\big)^{m+1} F_n^{p(m+1)}}}{n^m}, \\ \mathcal{L}(n; m, p) &= \frac{\sqrt[n]{\big((2n-1)!!\big)^{m+1} L_n^{p(m+1)}}}{n^m}. \end{aligned}$$

Then, it suffices to evaluate the limits

$$\begin{aligned} f(m, p) &= \lim_{n\to\infty} [\mathcal{F}(n+1; m, p) - \mathcal{F}(n; m, p)], \\ l(m, p) &= \lim_{n\to\infty} [\mathcal{L}(n+1; m, p) - \mathcal{L}(n; m, p)]. \end{aligned}$$

We claim the following

$$f(m, p) = l(m, p) = \left(\frac{2\alpha^p}{e}\right)^{m+1}. \tag{3}$$

Using the Stirling approximation for the factorial as well as the definition of the double factorial, we have the following:

$$(2n-1)!! = \frac{(2n)!}{2^n\, n!} = \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\big(1 + O(n^{-1})\big)}{2^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\big(1 + O(n^{-1})\big)} = \sqrt{2}\left(\frac{2n}{e}\right)^n \big(1 + O(n^{-1})\big).$$

Together with the Binet's formula for F_n, we find

$$\begin{aligned}\mathcal{F}(n;m,p) &= n^{-m}\left(\frac{2n}{e}\right)^{m+1}\left(\frac{\alpha^n-\beta^n}{\sqrt{5}}\right)^{\frac{p(m+1)}{n}}\left(1+O(n^{-1})\right)\\ &= n\left(\frac{2\alpha^p}{e}\right)^{m+1}\left(1+O(n^{-1})\right).\end{aligned}$$

Similarly, we find

$$\begin{aligned}\mathcal{L}(n;m,p) &= n^{-m}\left(\frac{2n}{e}\right)^{m+1}(\alpha^n+\beta^n)^{\frac{p(m+1)}{n}}\left(1+O(n^{-1})\right)\\ &= n\left(\frac{2\alpha^p}{e}\right)^{m+1}\left(1+O(n^{-1})\right).\end{aligned}$$

Thus, we obtain

$$f(m,p)=l(m,p)=\lim_{n\to\infty}(n+1-n)\left(\frac{2\alpha^p}{e}\right)^{m+1}\left(1+O(n^{-1})\right)=\left(\frac{2\alpha^p}{e}\right)^{m+1},$$

proving (3).

Also solved by Kenny B. Davenport, Dmitry Fleischman, Kevin Darío López Rodríguez, Raphael Schumacher, and the proposer.

Another Hassenberg Matrix Problem

B-1230 Proposed by T. Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all integers $n \geq 0$, prove that

$$F_{2n+1}=(-1)^n\sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}}(-1)^{t_1+t_3+\cdots+t_{n-[1+(-1)^n]/2}}\frac{(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!}\cdot 2^{t_1}.$$

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

We consider the Hassenberg matrix

$$H_n=\begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0\\ a_2 & a_1 & a_0 & \cdots & 0 & 0\\ a_3 & a_2 & a_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0\\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1\end{pmatrix}, \qquad a_0\neq 0.$$

It is known that

$$\det(H_n)=\sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}}(-a_0)^{n-(t_1+t_2+\cdots+t_n)}\frac{(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!}\,a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n}.$$

Let $a_0=1$, $a_1=2$,

$$a_2=a_4=\cdots=a_{2k}=-1,\quad 2k\leq n,\qquad \text{and}\qquad a_3=a_5=\cdots=a_{2k+1}=1,\quad 2k+1\leq n.$$

Then, we obtain

$$\begin{aligned} S_n &= (-1)^n \sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{t_1+t_3+\cdots+t_{n-[1+(-1)^n]/2}} \frac{(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!}\cdot 2^{t_1} \\ &= \det\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & 1 & \cdots & 0 & 0 \\ 1 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-2} & (-1)^{n-3} & (-1)^{n-4} & \cdots & 2 & 1 \\ (-1)^{n-1} & (-1)^{n-2} & (-1)^{n-3} & \cdots & -1 & 2 \end{pmatrix}. \end{aligned}$$

Adding row $k-1$ to row k in this matrix for $k=n$ to $k=2$, we get that this determinant equals

$$\det\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix} = 2\det(A_{n-1}) - \det(A_{n-2}),$$

where

$$A_n = \begin{pmatrix} 3 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix}_{n\times n}.$$

It is an easy exercise to find the recurrence relation $\det(A_n) = 3\det(A_{n-1}) - \det(A_{n-2})$, from which we determine that $\det(A_n) = F_{2n+2}$. Therefore,

$$S_n = 2F_{2n} - F_{2n-2} = F_{2n+1}.$$

Editor's Note: Both Rodríguez and Stadler noticed that the generating function for F_{2n+1} is $\frac{1-x}{1-3x+x^2}$, which is in the form of a composite function $f(g(x))$, where $f(x) = \frac{1}{1+x}$ and $g(x) = \frac{x^2-2x}{1-x}$. Consequently, they were able to derive the desired result by applying the Faá di Bruno formula for the derivative of $f(g(x))$. See the solution to Elementary Problem B-1210 that appeared in Volume 56.2 (2018), pages 183–184.

Also solved by Kevin Darío López Rodríguez, Albert Stadler, and the proposer.