ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2015. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1151 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Calculate each of the following:

(i) $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n}\right)$,

(ii) $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!L_{n+1}} - \sqrt[n]{n!L_n}\right)$,

(iii) $\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!F_{n+1}} - \sqrt[n]{(2n-1)!!F_n}\right)$,

(iv) $\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!L_{n+1}} - \sqrt[n]{(2n-1)!!L_n}\right)$.

B-1152 **Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.**

For any positive integer number k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recurrently by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$ with respective initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. Find a closed form expression for

$$\sum_{n=1}^{\infty} \frac{F_{k,2^n}}{1 + L_{k,2^{n+1}}}$$

as a function of $\alpha = \frac{k+\sqrt{k^2+4}}{2}$.

B-1153 **Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.**

For any positive integer number k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recurrently by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$ with respective initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. Prove that

$$\sum_{i=0}^{n} \left(\frac{2}{k}\right)^i L_{k,i} = k \left(\frac{2}{k}\right)^{n+1} F_{k,n+1}.$$

B-1154 **Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA.**

Find a closed form expression for

$$\sum_{i=0}^{n} L_i^2 L_{i+1}^2.$$

B-1155 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Prove each of the following:

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{L_{2^{k+1}}}{F_{3\cdot 2^k}} = \frac{5}{4}$,

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{F_{2^{k-1}}^2}{L_{2^k}^2 - 1} = \frac{3}{20}$.

SOLUTIONS

Tedious But Pretty Identities

B-1131 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 51.3, August 2013)

For integers a and b, prove that

$$(F_a^2 + F_{a+1}^2 + F_{a+2}^2)(F_aF_b + F_{a+1}F_{b+1} + F_{a+2}F_{b+2}) \\ = 2(F_a^3F_b + F_{a+1}^3F_{b+1} + F_{a+2}^3F_{b+2}); \qquad (1)$$

and

$$(F_a^2 + F_{a+1}^2 + F_{a+2}^2)(F_aF_b^3 + F_{a+1}F_{b+1}^3 + F_{a+2}F_{b+2}^3) \\ = (F_b^2 + F_{b+1}^2 + F_{b+2}^2)(F_a^3F_b + F_{a+1}^3F_{b+1} + F_{a+2}^3F_{b+2}). \qquad (2)$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For convenience, let x, y, z denote F_a, F_{a+1}, F_{a+2}, and u, v, w denote F_b, F_{b+1}, F_{b+2}, respectively. Using the equalities $z = x + y$ and $w = u + v$ we have

$$\begin{aligned} \text{LHS} &= 2(x^2 + y^2 + xy)(xu + yv + (x + y)(u + v)) \\ &= 2(2ux^3 + vx^3 + 3ux^2y + 3vx^2y + 3uxy^2 + 3vxy^2 + uy^3 + 2vy^3) \\ &= 2(ux^3 + vy^3 + ux^3 + vx^3 + 3ux^2y + 3vx^2y + 3uxy^2 + 3vxy^2 + uy^3 + vy^3) \\ &= 2(x^3u + y^3v + (x + y)^3(u + v) \\ &= \text{RHS}. \end{aligned}$$

In order to prove (2) it is enough to prove that

$$(x^2 + y^2 + xy)(xu^3 + yv^3 + (x + y)(u + v)^3) = (u^2 + v^2 + uv)(x^3u + y^3v + (x + y)^3(u + v)).$$

The expansion of both terms yields the same answer as can be checked with the aid of a software package like *Mathematica* or *MATLAB*.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Esref Gurel and Mustafa Asci (jointly), Russell J. Hendel, and the proposer.

The AM-GM Inequality Paves the Way

B-1132 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania.
(Vol. 51.3, August 2013)

Prove that

$$\text{(i)} \quad \mathrm{F}_n^2 + \mathrm{F}_{n+1}^2 + 5\mathrm{F}_{n+2}^2 > 4\sqrt{6}\sqrt{\mathrm{F}_n\mathrm{F}_{n+1}}\mathrm{F}_{n+2},$$

$$\text{(ii)} \quad \mathrm{L}_n^2 + \mathrm{L}_{n+1}^2 + 5\mathrm{L}_{n+2}^2 > 4\sqrt{6}\sqrt{\mathrm{L}_n\mathrm{L}_{n+1}}\mathrm{L}_{n+2},$$

for any positive integer n.

Solution by Robinson Higuita (student), Universidad de Antioquia, Columbia.

(i) From

$$F_{n+4}^2 = (F_{n+2} + F_{n+3})^2 = (2F_n + 3F_{n+1})^2 = ((2F_n)^2 + (3F_{n+1})^2) + 12F_nF_{n+1}$$

and the inequality of arithmetic and geometric means we have that

$$F_{n+4}^2 \geq 12F_nF_{n+1} + 12F_nF_{n+1} = 24F_nF_{n+1}.$$

That is, $2\sqrt{6F_nF_{n+1}} \leq F_{n+4}$. Therefore,

$$4\sqrt{6}\sqrt{F_nF_{n+1}}F_{n+2} \leq 2F_{n+4}F_{n+2} = 2(2F_{n+2} + F_{n+1})F_{n+2} = 4F_{n+2}^2 + 2F_{n+1}F_{n+2}.$$

This and the inequality of arithmetic and geometric mean imply $2F_{n+1}F_{n+2} \leq F_n^2 + F_{n+1}^2$. So,

$$4\sqrt{6}\sqrt{F_nF_{n+1}}F_{n+2} \leq 4F_{n+2}^2 + 2F_{n+1}F_{n+2} < 5F_{n+2}^2 + F_n^2 + F_{n+1}^2.$$

(ii) The proof of

$$4\sqrt{6}\sqrt{L_nL_{n+1}}L_{n+2} < 5L_{n+2}^2 + L_n^2 + L_{n+1}^2$$

is similar to the previous proof. It is enough to replace F_n with L_n.

Also solved by Kenneth B. Davenport, Dmitry Freischman, Esref Gurel and Mustafa Asci (jointly), Russell Jay Hendel, Zbigniew Jakubczyk, and the proposer.

The Value of a Series of Reciprocal Fibonacci Numbers

B-1133 Proposed by Mohammed K. Azarian, University of Evansville, Indiana. (Vol. 51.3, August 2013)

Determine the value of the following infinite series

$$S = \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{3\cdot 8} - \frac{1}{5\cdot 8} + \frac{1}{8\cdot 13} + \frac{1}{8\cdot 21} - \frac{1}{13\cdot 21} + \frac{1}{21\cdot 34} + \frac{1}{21\cdot 55} - \frac{1}{34\cdot 55} + \frac{1}{55\cdot 89} + \frac{1}{55\cdot 144} - \frac{1}{89\cdot 144} + \frac{1}{144\cdot 233} + \frac{1}{144\cdot 377} - \cdots.$$

Solution by John D. Watson, Jr. (student), The Citadel, The Military College of South Carolina.

We start by expressing the original problem in terms of a series

$$S = \sum_{k=1}^{\infty} = \frac{1}{F_{2k}F_{2k+1}} + \frac{1}{F_{2k}F_{2k+2}} - \frac{1}{F_{2k+1}F_{2k+2}}.$$

Simplifying:

$$\begin{aligned}
S &= \sum_{k=1}^{\infty} \frac{F_{2k+2} + F_{2k+1} - F_{2k}}{F_{2k}F_{2k+1}F_{2k+2}} \\
&= \sum_{k=1}^{\infty} \frac{2F_{2k+1}}{F_{2k}F_{2k+1}F_{2k+2}} \\
&= \sum_{k=1}^{\infty} \frac{2}{F_{2k}F_{2k+2}} \\
&= 2\sum_{k=1}^{\infty} \frac{1}{F_{2k}F_{2k+1}} - \frac{1}{F_{2k+1}F_{2k+2}} \\
&= 2\sum_{i=2}^{\infty} \frac{(-1)^i}{F_iF_{i+1}}.
\end{aligned}$$

Let S_n be the nth partial sum of this series. Since S_n is a telescoping sum, we have that

$$S_n = 2\sum_{i=2}^{n} \frac{(-1)^i}{F_iF_{i+1}} = 2\left(\frac{F_1}{F_2} - \frac{F_n}{F_{n+1}}\right).$$

Therefore,

$$S = 2\lim_{n\to\infty}\left(\frac{F_1}{F_2} - \frac{F_n}{F_{n+1}}\right) = 2\left(1 - \frac{1}{\alpha}\right) = \frac{-2\beta}{\alpha} = 3 - \sqrt{5}.$$

Also solved by Brian D. Beasley, Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, Robinson Higuita and Bilson Castro (jointly), Ángel Plaza, and the proposer.

Easier Than It Looks

B-1134 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain and Francesc Gispert Sánchez, CFIS, Barcelona Tech, Barcelona, Spain. (Vol. 51.3, August 2013)

Let n be a positive integer. Prove that

$$\frac{1}{F_nF_{n+1}}\left[\left(1 - \frac{1}{n}\right)\sum_{k=1}^{n} F_k^{2n} + \prod_{k=1}^{n} F_k^2\right] \geq \left(\prod_{k=1}^{n} F_k^{(1-1/n)}\right)^2.$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the AM-GM inequality, the LHS of the given inequality satisfies

$$\begin{aligned}
\text{LHS} &\geq \frac{1}{F_nF_{n+1}}\left[(n-1)\sqrt[n]{\prod_{k=1}^{n}F_k^{2n}}+\prod_{k=1}^{n}F_k^2\right]\\
&= \frac{1}{F_nF_{n+1}}\left[(n-1)\prod_{k=1}^{n}F_k^2+\prod_{k=1}^{n}F_k^2\right]\\
&= \frac{1}{F_nF_{n+1}}\left[n\prod_{k=1}^{n}F_k^2\right]\\
&= \frac{n\prod_{k=1}^{n}F_k^2}{\sum_{k=1}^{n}F_i^2} \qquad \left(\text{since } F_nF_{n+1}=\sum_{i=1}^{n}F_i^2\right)\\
&\geq \frac{\prod_{k=1}^{n}F_k^2}{\sqrt[n]{\prod_{k=1}^{n}F_k^2}} \qquad (\text{by the AM-GM inequality})\\
&= \text{RHS}.
\end{aligned}$$

Also solved by Dmitry Fleischman and the proposer.

A General Inequality Applied to Fibonacci and Lucas Numbers

B-1135 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.
(Vol. 51.3, August 2013)

Prove that

$$\frac{F_{n+1}^2}{F_n^3(F_nF_{n+1}+F_{n+2}^2)}+\frac{F_{n+2}^2}{F_{n+1}^3(F_{n+1}F_{n+2}+F_n^2)}+\frac{F_n^2}{F_{n+2}^3(F_nF_{n+2}+F_{n+1}^2)}>\frac{3}{2F_nF_{n+1}F_{n+2}}; \quad (1)$$

and

$$\frac{L_{n+1}^2}{L_n^3(L_nL_{n+1}+L_{n+2}^2)}+\frac{L_{n+2}^2}{L_{n+1}^3(L_{n+1}L_{n+2}+L_n^2)}+\frac{L_n^2}{L_{n+2}^3(L_nL_{n+2}+L_{n+1}^2)}>\frac{3}{2L_nL_{n+1}L_{n+2}}; \quad (2)$$

for any positive integer n.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The proposed inequalities are particular cases of the following more general inequality for positive numbers x, y, z:

$$\frac{x^2}{z^3(xz+y^2)}+\frac{y^2}{x^3(xy+z^2)}+\frac{z^2}{y^3(yz+x^2)}\geq\frac{3}{2xyz},$$

which may be written as

$$\frac{x^3y}{z^2(xz+y^2)}+\frac{y^3z}{x^2(xy+z^2)}+\frac{z^3x}{y^2(yz+x^2)}\geq\frac{3}{2}$$
$$\frac{\frac{x^2}{z^2}}{\frac{z}{y}+\frac{y}{x}}+\frac{\frac{y^2}{x^2}}{\frac{x}{z}+\frac{z}{y}}\geq\frac{3}{2}.$$

By changing variables $a=\frac{x}{z}$, $b=\frac{z}{y}$, and $c=\frac{y}{x}$, the last inequality reads

$$\frac{a^2}{b+c}+\frac{b^2}{a+c}+\frac{c^2}{a+b}\geq\frac{3}{2}. \tag{3}$$

Then, by Chebyshev's sum inequality, the LHS of (3) satisfies

$$\begin{aligned}\text{LHS} &\geq \frac{a+b+c}{3}\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)\\ &\geq \sqrt[3]{abc}\cdot\frac{3}{2} \qquad \text{(by the AM-GM and Nesbitt inequalities)}\\ &= \frac{3}{2},\end{aligned}$$

since $abc=1$.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, and the proposer.

We would like to acknowledge Dmitry Fleischman for solving B-1127.