ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at JAWADS@nwmissouri.edu.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2017. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1191 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For nonnegative integers m and n, prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{L_{mk}}{L_m^k} = \frac{L_{mn}}{L_m^n}.$$

B-1192 Proposed by T. Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let M_n be an $n \times n$ matrix given for all $n \geq 1$ by

$$M_n = \begin{pmatrix} F_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ F_2 & F_1 & 1 & \dots & 0 & 0 & 0 \\ F_3 & F_2 & F_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ F_{n-1} & F_{n-2} & F_{n-3} & \dots & F_2 & F_1 & 1 \\ F_n & F_{n-1} & F_{n-2} & \dots & F_3 & F_2 & F_1 \end{pmatrix}.$$

Prove that

$$\det(M_n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

B-1193 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

If $F_1^2, F_2^2, \dots, F_n^2$ are the square of the first n Fibonacci numbers, then find real numbers $a_1, a_2, \dots, a_n$ satisfying $a_k > F_k^2$, $1 \leq k \leq n$, and

$$\frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} a_k < \frac{\alpha^2}{\alpha - 1}.$$

B-1194 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that

$$\frac{L_1}{(L_1^2 + L_2^2 + 2)^{m+1}} + \frac{L_2}{(L_1^2 + L_2^2 + L_3^2 + 2)^{m+1}} + \cdots + \frac{L_n}{(L_1^2 + L_2^2 + \cdots + L_{n+1}^2 + 2)^{m+1}}$$
$$\geq \frac{(L_{n+2} - 1)^{m+1}}{L_{n+2}^{m+1}(L_{n+2} - 3)^m}$$

for any positive integers n and m.

B-1195 Proposed by Jeremiah Bartz, Francis Marion University, Florence, SC.

Let G_i denote the generalized Fibonacci sequence given by $G_1 = a$, $G_2 = b$, and $G_i = G_{i-1} + G_{i-2}$ for $i \geq 3$. Let $m \geq 1$ and $k \geq 0$. Prove that the area A of the polygon with $n \geq 3$ vertices

$$(G_m, G_{m+k}), (G_{m+2k}, G_{m+3k}), \dots, (G_{m+(2n-2)k}, G_{m+(2n-1)k})$$

is

$$\frac{|\mu| F_k (F_{2k(n-1)} - (n-1) F_{2k})}{2}$$

where $\mu = a^2 + ab - b^2$.

SOLUTIONS

All is One!

B-1171 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 53.3, August 2015)

For all integers $n \geq 1$, compute

$$\frac{(F_{n-1}+F_{n+1})^3+(2F_n+F_{n+3})^3+(5F_n+F_{n+3})^3+(9F_n+F_{n+3})^3}{8F_{n+1}^3+F_{n+3}^3+(7F_n+F_{n+3})^3+(8F_n+F_{n+3})^3}.$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC.

For each integer $n \geq 1$, we have $F_{n+3} = 2F_{n+1} + F_n$ and $F_{n-1} = F_{n+1} - F_n$. Let $x = F_n$ and $y = F_{n+1}$. The the numerator of the given fraction is

$$(-x+2y)^3+(3x+2y)^3+(6x+2y)^3+(10x+2y)^3 = 1242x^3+876x^2y+216xy^2+32y^3$$

and the denominator is

$$8y^3+(x+2y)^3+(8x+2y)^3+(9x+2y)^3 = 1242x^3+876x^2y+216xy^2+32y^3.$$

Hence, the given fraction equals 1.

All received solutions were similar to the above. Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, G. C. Greubel, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Nicuşor Zlota, and the proposer.

Area of Triangles with Generalized Fibonacci Number Coordinates

B-1172 Proposed by Steve Edwards, Kennesaw State University, Marietta, GA. (Vol. 53.3, August 2015)

Show that the area of the triangle whose vertices have coordinates (F_n, F_{n+k}), (F_{n+2k}, F_{n+3k}), (F_{n+4k}, F_{n+5k}) is

$$\frac{5F_k^4L_k}{2} \text{ if } k \text{ is even and } \frac{F_k^2L_k^3}{2} \text{ if } k \text{ is odd.}$$

Also, find the area of the triangle whose vertices have coordinates (L_n, L_{n+k}), (L_{n+2k}, L_{n+3k}), (L_{n+4k}, L_{n+5k}).

Solution by Virginia P. Johnson, Columbia College, Columbia, South Carolina.

Consider the triangle with vertices given by Fibonacci numbers as follows: (F_n, F_{n+k}), (F_{n+2k}, F_{n+3k}), and (F_{n+4k}, F_{n+5k}). The (signed) area of this triangle can be calculated using the determinant formula from vector calculus:

$$\begin{aligned} A &= \frac{1}{2}\begin{vmatrix} F_{n+2k}-F_n & F_{n+3k}-F_{n+k} \\ F_{n+4k}-F_n & F_{n+5k}-F_{n+k} \end{vmatrix} \\ &= \frac{1}{2}\begin{vmatrix} F_{n+2k}-F_n & F_{n+3k}-F_{n+k} \\ F_{n+4k}-F_{n+2k} & F_{n+5k}-F_{n+3k} \end{vmatrix}. \end{aligned}$$

Note that all entries in the determinant are of the form $F_{m+2k} - F_m$. It will be more convenient to find this determinant for the general case first. Let a, b, r be real numbers with $r \neq 0$. Define a real valued function on the integers by

$$f(n) = ar^n + b\frac{(-1)^{n+1}}{r^n}.$$

This is the general solution to the recursion

$$f(n) = pf(n-1) + f(n-2)$$

where p and r are related by

$$p = r - \frac{1}{r}$$

that is $r^2 - pr - 1 = 0$. For any real value of p, this has real solutions so that the method works for any recursion of this form.

Then for the general case, the signed area can be determined using

$$A = \frac{1}{2}\begin{vmatrix} f(n+2k) - f(n) & f(n+3k) - f(n+k) \\ f(n+4k) - f(n+2k) & f(n+5k) - f(n+3k) \end{vmatrix}.$$

All entries in the determinant are of the form $f(m+2k) - f(m)$.

$$\begin{aligned} f(m+2k) - f(m) &= ar^{m+2k} + b\frac{(-1)^{m+2k+1}}{r^{m+2k}} - \left(ar^m + b\frac{(-1)^{m+1}}{r^m}\right) \\ &= ar^m(r^{2k}-1) + (-1)^{m+1}b\left(\frac{1}{r^{m+2k}} - \frac{1}{r^m}\right) \\ &= ar^m(r^{2k}-1) + (-1)^m b\left(\frac{r^{2k}-1}{r^{m+2k}}\right) \\ &= (r^{2k}-1)\left(ar^m + b\frac{(-1)^m}{r^{m+2k}}\right) \\ &= (r^{2k}-1)\left(\frac{ar^{2m+2k} + (-1)^m b}{r^{m+2k}}\right). \end{aligned}$$

Each entry of the matrix will have a factor of $(r^{2k} - 1)$ which we can factor out of the two rows. The area for the general case is therefore

$$A = \frac{(r^{2k}-1)^2}{2}\begin{vmatrix} \frac{ar^{2n+2k}+(-1)^n b}{r^{n+2k}} & \frac{ar^{2n+4k}+(-1)^{n+k} b}{r^{n+3k}} \\ \frac{ar^{2n+6k}+(-1)^n b}{r^{n+4k}} & \frac{ar^{2n+8k}+(-1)^{n+k} b}{r^{n+5k}} \end{vmatrix}.$$

To clear the fractions we factor $\frac{1}{r^{n+3k}}$ out of the first row and $\frac{1}{r^{n+5k}}$ of the second row.

$$\begin{aligned} A &= \frac{(r^{2k}-1)^2}{2r^{n+3k}r^{n+5k}}\begin{vmatrix} ar^{2n+3k} + (-1)^n br^k & ar^{2n+4k} + (-1)^{n+k} b \\ ar^{2n+7k} + (-1)^n br^k & ar^{2n+8k} + (-1)^{n+k} b \end{vmatrix} \\ &= \frac{ab(-1)^n}{2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)\left(r^k + \frac{(-1)^{k+1}}{r^k}\right). \end{aligned}$$

If $a = b = \frac{1}{\sqrt{5}}$, and $r = \frac{1+\sqrt{5}}{2}$, then $f(n)$ reduces to the Fibonacci numbers, that is $f(n) = F_n$. To get the Lucas numbers let $\overline{a} = 1$, $\overline{b} = -1$, and $r = \frac{1+\sqrt{5}}{2}$

$$k \text{ is even} \Rightarrow \begin{cases} a\left(r^k - \frac{1}{r^k}\right) = a\left(r^k + \frac{(-1)^{k+1}}{r^k}\right) = F_k, \\ \left(r^k + \frac{1}{r^k}\right) = \left(r^k + \frac{(-1)(-1)^{k+1}}{r^k}\right) = L_k, \end{cases}$$

and

$$k \text{ is odd} \Rightarrow \begin{cases} a\left(r^k + \frac{1}{r^k}\right) = a\left(r^k + \frac{-1(-1)^{k+1}}{r^k}\right) = F_k, \\ \left(r^k - \frac{1}{r^k}\right) = \left(r^k + \frac{-1(-1)^{k+1}}{r^k}\right) = L_k. \end{cases}$$

Fibonacci Numbers: Then $a = b = \frac{1}{\sqrt{5}}$. We can now calculate the signed area for a triangle with vertices given by Fibonacci numbers (F_n, F_{n+k}), (F_{n+2k}, F_{n+3k}), and (F_{n+4k}, F_{n+5k}).

If k is even, the signed area of the triangle will be

$$\begin{aligned} A &= \frac{ab}{2}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right) \\ &= \frac{a^4}{2a^2}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right) \\ &= \frac{5}{2}F_k^4 L_k. \end{aligned}$$

If k is odd, the signed area of the triangle will be

$$\begin{aligned} A &= \frac{a^2}{2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)^2 \\ &= \frac{F_k^2 L_k^3}{2}. \end{aligned}$$

Lucas Numbers: Similarly, if a triangle with vertices given by Lucas numbers, (L_n, L_{n+k}), (L_{n+2k}, L_{n+3k}), and (L_{n+4k}, L_{n+5k}) and using that in this case $\overline{a} = 1$, $\overline{b} = -1$, the signed area will be as follows.

If k is even:

$$\begin{aligned} A &= \frac{\overline{a}\overline{b}}{2}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right) \\ &= \frac{-a^4}{2a^4}\left(r^k + \frac{(-1)^{k+1}}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right). \end{aligned}$$

The absolute value gives the area as $\frac{25}{2}F_k^4 L_k$.

If k is odd:

$$\begin{aligned} A &= \frac{\overline{a}\overline{b}}{2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)^2 \\ &= \frac{-a^2}{2a^2}\left(r^k + \frac{1}{r^k}\right)^2\left(r^k - \frac{1}{r^k}\right)^3. \end{aligned}$$

The absolute value gives the area as $\frac{5F_k^2 L_k^3}{2}$.

Using appropriate value for a, b, and r, it is possible to calculate the area for triangles when Generalized Fibonacci Numbers, Pell Numbers, or Pell-Lucas Numbers are used as described for the vertices.

The featured solution, though not the shortest, was selected for its generalized treatment of the problem. Also solved by Jeremiah Bartz, Charles K. Cook, Ravi

Kumar Davala, G. C. Greubel, Harris Kwong, Ángel Plaza, and the proposer. A nameless solution was received.

By Induction

B-1173 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 53.3, August 2015)

(i) Prove that

$$\frac{F_1}{(F_1^2+F_2^2)^{m+1}}+\frac{F_2}{(F_1^2+F_2^2+F_3^2)^{m+1}}+\cdots+\frac{F_n}{(F_1^2+F_2^2+\cdots+F_{n+1}^2)^{m+1}}\geq\frac{1}{F_{n+2}^m}-\frac{1}{F_{n+2}^{m+1}}$$

for any positive integer n and any positive real number m.

(ii) (Corrected) Prove that

$$\frac{L_1}{(L_1^2+L_2^2+2)^2}+\frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^2}+\cdots+\frac{L_n}{(L_1^2+L_2^2+\cdots+L_{n+1}^2+2)^2}\geq\frac{L_{n+2}-3}{L_{n+2}^2(L_{n+2}-1)^2}$$

for any positive integer n.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

We use $\sum_{k=1}^n F_k^2=F_nF_{n+1}$ and $\sum_{k=1}^n L_k^2=L_nL_{n+1}-2$.

(i) The LHS is

$$\frac{F_1}{(F_2F_3)^{m+1}}+\frac{F_2}{(F_3F_4)^{m+1}}+\cdots+\frac{F_n}{(F_{n+1}F_{n+2})^{m+1}}$$

while the RHS is

$$\frac{F_{n+2}-1}{F_{n+2}^{m+1}}=\frac{\sum_{k=1}^n F_k}{F_{n+2}^{m+1}}.$$

We prove that

$$\frac{F_1}{(F_2F_3)^{m+1}}+\frac{F_2}{(F_3F_4)^{m+1}}+\cdots+\frac{F_n}{(F_{n+1}F_{n+2})^{m+1}}\geq\frac{F_{n+2}-1}{F_{n+2}^{m+1}}.$$

This may be done by induction.

For $n=1$ the inequality becomes

$$\frac{F_1}{(F_2F_3)^{m+1}}\geq\frac{F_3-1}{F_3^{m+1}},$$

that is $\frac{1}{2^{m+1}}\geq\frac{1}{2^{m+1}}$, which is true. Let us suppose the inequality holds for $n>1$. Then for $n+1$, by induction hypothesis we have

$$\frac{F_1}{(F_2F_3)^{m+1}}+\cdots+\frac{F_n}{(F_{n+1}F_{n+2})^{m+1}}+\frac{F_{n+1}}{(F_{n+2}F_{n+3})^{m+1}}\geq\frac{F_{n+2}-1}{F_{n+2}^{m+1}}+\frac{F_{n+1}}{(F_{n+2}F_{n+3})^{m+1}}.$$

It is enough to see that

$$\frac{F_{n+2}-1}{F_{n+2}^{m+1}}+\frac{F_{n+1}}{(F_{n+2}F_{n+3})^{m+1}}\geq\frac{F_{n+3}-1}{F_{n+3}^{m+1}}$$

$$F_{n+3}^{m+1}(F_{n+2}-1)+F_{n+1}\geq F_{n+2}^{m+1}(F_{n+3}-1)$$

which is true, since there is an identity for $m=0$.
(ii) The RHS of this inequality should be $\frac{L_{n+2}-3}{L_{n+2}^2(L_{n+2}-1)^2}$ as it will be shown below.

The LHS is

$$\frac{L_1}{(L_2L_3)^2}+\frac{L_2}{(L_3L_4)^2}+\cdots+\frac{L_n}{(L_{n+1}L_{n+2})^2},$$

so we have to prove that

$$\frac{L_1}{(L_2L_3)^2}+\frac{L_2}{(L_3L_4)^2}+\cdots+\frac{L_n}{(L_{n+1}L_{n+2})^2}\geq\frac{L_{n+2}-3}{L_{n+2}^2(L_{n+2}-1)^2},$$

and this may be done by induction.

For $n=1$ the inequality becomes

$$\frac{L_1}{(L_2L_3)^2}\geq\frac{L_3-3}{L_3^2(L_3-1)^2},$$

that is $\frac{1}{12^2}\geq\frac{1}{4^23^2}$, which is true. Let us suppose the inequality holds for $n>1$. Then for $n+1$, by induction hypothesis we have

$$\frac{L_1}{(L_2L_3)^2}+\cdots+\frac{L_n}{(L_{n+1}L_{n+2})^2}+\frac{L_{n+1}}{(L_{n+2}L_{n+3})^2}\geq\frac{L_{n+2}-3}{L_{n+2}^2(L_{n+2}-1)^2}+\frac{L_{n+1}}{(L_{n+2}L_{n+3})^2},$$

so it is enough to see that

$$\frac{L_{n+2}-3}{L_{n+2}^2(L_{n+2}-1)^2}+\frac{L_{n+1}}{(L_{n+2}L_{n+3})^2}\geq\frac{L_{n+3}-3}{L_{n+3}^2(L_{n+3}-1)^2}:$$

$$\begin{aligned}\frac{L_{n+2}-3}{L_{n+2}^2(L_{n+2}-1)^2}+\frac{L_{n+1}}{(L_{n+2}L_{n+3})^2}&\geq\frac{L_{n+2}-3}{L_{n+3}^2(L_{n+3}-1)^2}+\frac{L_{n+1}}{(L_{n+3}-1)^2L_{n+3}^2}\\&=\frac{L_{n+3}-3}{L_{n+3}^2(L_{n+3}-1)^2}.\end{aligned}$$

Also solved by Kenneth B. Davenport and the proposer.

A Summation of Recipricals

**B-1174 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 53.3, August 2015)**

Prove that

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{F_n^4-1}=-\frac{1}{18}.$$

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

Using the identities $F_n^4-1=F_{n-2}F_{n-1}F_{n+1}F_{n+2}$ and $F_{n+2}F_{n-2}-F_{n-1}F_{n+2}=2(-1)^{n+1}$ (see [1]), we have

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{(-1)^n}{F_n^4-1}=\sum_{n=3}^{\infty}\frac{(-1)^n}{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}&=-\frac{1}{2}\sum_{n=3}^{\infty}\frac{F_{n+2}F_{n-2}-F_{n+1}F_{n-1}}{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}\\&=-\frac{1}{2}\sum_{n=3}^{\infty}\left[\frac{1}{F_{n-1}F_{n+1}}-\frac{1}{F_{n-2}F_{n+2}}\right].\end{aligned}$$

According to Problem B-9 of The Fibonacci Quarterly,

$$\sum_{n=3}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = \frac{1}{2}.$$

Using the identity $F_{n+2}F_{n-3} - F_{n+1}F_{n-2} = 3(-1)^n$, [1] we have

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{1}{F_{n-2}F_{n+2}} &= \frac{1}{3}\sum_{n=3}^{\infty}(-1)^n \frac{F_{n+2}F_{n-3} - F_{n+1}F_{n-2}}{F_{n-2}F_{n+2}} \\ &= \frac{1}{3}\sum_{n=3}^{\infty}(-1)^n \left[\frac{F_{n-3}}{F_{n-2}} - \frac{F_{n+1}}{F_{n+2}}\right].\end{aligned}$$

This last series is telescoping with the nth partial sum

$$-\frac{F_0}{F_1} + \frac{F_1}{F_2} - \frac{F_2}{F_3} + \frac{F_3}{F_4} - \frac{F_{n-1}}{F_n} + \frac{F_n}{F_{n+1}} - \frac{F_{n+1}}{F_{n+2}} + \frac{F_{n+2}}{F_{n+3}}.$$

Since $\lim_{n\to m} \frac{F_{n+1}}{F_n} = \alpha$, this last series converges to $1 - \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$, so the original series converges to

$$-\frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\cdot\frac{7}{6}\right) = -\frac{1}{18}.$$

References

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Halsted Press, (1989).

Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, Ángel Plaza, and the proposer.

Based on the AM-GM or PM Inequality

B-1175 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 53.3, August 2015)

Let $m \geq 0$ and $n \in \mathbb{N}$. Prove that $\left(\sqrt{F_{2n+1}} - F_{n+1}\right)^m + \left(\sqrt{F_{2n+1}} + F_{n+1}\right)^m \geq 2F_n^m$.

All solvers submitted more or less the same solution as the following one.

Since $F_{2n+1} = F_n^2 + F_{n+1}^2$, [1, (11)] the conclusion follows by the AM-GM inequality. In fact,

$$\begin{aligned}&\frac{\left(\sqrt{F_n^2 + F_{n+1}^2} - F_{n+1}\right)^m + \left(\sqrt{F_n^2 + F_{n+1}^2} + F_{n+1}\right)^m}{2} \\ &\geq \sqrt{\left(\sqrt{F_n^2 + F_{n+1}^2} - F_{n+1}\right)^m \left(\sqrt{F_n^2 + F_{n+1}^2} + F_{n+1}\right)^m} \\ &= (F_n^2 + F_{n+1}^2 - F_{n+1}^2)^{m/2} \\ &= F_n^m.\end{aligned}$$

References

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, (2008).

Also solved by Brian Bradie, G. C. Greubel, Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Nicuşor Zlota, and the proposer.