ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2016. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1181 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{(F_n^2+1)^2} = \frac{5}{3}.$$

B-1182 **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.**

Let n be a positive integer. If a, b, c are the roots of the equation $x^3 - F_n x^2 + F_{n+1} = 0$, then prove that

$$a^3(F_n - a) + b^3(F_n - b) + c^3(F_n - c)$$

is a positive integer which is the sum of n squares.

B-1183 **Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

Express

$$\sum_{n=k}^{\infty} \binom{n}{k} \frac{2}{L_{n+1} + \sqrt{5}F_{n+1}}$$

as a function of F_{k+1} and L_{k+1}.

B-1184 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.**

Let k be a positive integer.

(1) If $A(k) = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$, compute $\displaystyle\prod_{k=1}^{n} A(k)$.

(2) If $B(k) = \begin{pmatrix} F_k^2 & F_{k+1}^2 \\ F_{k+1}^2 & F_k^2 \end{pmatrix}$, compute $\displaystyle\prod_{k=1}^{n} B(k)$.

(3) If $C(k) = \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \begin{pmatrix} L_k & L_{k+1} \\ L_{k+1} & L_k \end{pmatrix}$, compute $\displaystyle\prod_{k=1}^{n} C(k)$.

B-1185 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.**

Let a and b be positive real numbers such that $aF_{n+1} > bF_n$. Prove that

$$\sum_{k=1}^{n} \frac{F_k^2}{aF_nF_{n+1} - bF_k^2} \geq \frac{n}{an - b}$$

for all $n \geq 1$.

SOLUTIONS

Powered Index Series and Inverse Tangent of an Inverse Series

B-1161 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 53.1, February 2015)

Prove each of the following:

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k L_{2^k}}{F_{2^k}^2} = 10,$

(ii) $\displaystyle\sum_{k=0}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{5}F_{2k+1}}\right) = \frac{\pi}{4}.$

Solution by Zbigniew Jakubczyk, Warsaw, Poland.

(i) We will make use of the identities $F_{2n} = F_n L_n$ and $L_{2n} = L_n^2 - 2(-1)^n$. If k is an integer greater than 2, then

$$\frac{2^k L_{2^k}}{F_{2^k}^2} = \frac{2^k(L_{2^{k-1}}^2 - 2)}{F_{2^k}^2} = \frac{2^k L_{2^{k-1}}^2 - 2^{k+1}}{F_{2^k}^2} = \frac{2^k L_{2^{k-1}}^2}{F_{2^{k-1}}^2 L_{2^{k-1}}^2} - \frac{2^{k+1}}{F_{2^k}^2} = \frac{2^k}{F_{2^{k-1}}^2} - \frac{2^{k+1}}{F_{2^k}^2}.$$

So,

$$\sum_{k=1}^{\infty} \frac{2^k L_{2^k}}{F_{2^k}^2} = \lim_{n\to\infty}\left(\frac{2L_2}{F_2^2} + \sum_{k=2}^{n}\left(\frac{2^k}{F_{2^{k-1}}^2} - \frac{2^{k+1}}{F_{2^k}^2}\right)\right) = \lim_{n\to\infty}\left(\frac{2L_2}{F_2^2} + \frac{2^2}{F_1^2} - \frac{2^{n+1}}{F_{2^n}^2}\right) = 10.$$

(ii) Making use of the results $F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}$, $\alpha\beta = -1$, $\alpha + \beta = 1$, $\tan^{-1}\frac{y-x}{1+xy} = \tan^{-1} y - \tan^{-1} x$, and $\lim_{n\to\infty}\beta^n = 0$, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{5}F_{2k+1}}\right) &= \lim_{n\to\infty}\sum_{k=0}^{n} \tan^{-1}\left(\frac{1}{\alpha^{2k+1} - \beta^{2k+1}}\right) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} \tan^{-1}\left(\frac{1}{-\frac{1}{\beta^{2k+1}} - \beta^{2k+1}}\right) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} \tan^{-1}\left(\frac{-\beta^{2k+1}}{1 + \beta^{4k+2}}\right) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} \tan^{-1}\left(\frac{(\frac{1}{\beta} - \beta)\beta^{2k+1}}{1 + \beta^{4k+2}}\right) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} \tan^{-1}\frac{\beta^{2k} - \beta^{2k+2}}{1 + \beta^{2k}\beta^{2k+2}} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}(\tan^{-1}(\beta^{2k}) - \tan^{-1}(\beta^{2k+2})) \\
&= \lim_{n\to\infty}(\tan^{-1}(\beta^{0}) - \tan^{-1}(\beta^{2n+2})) = \frac{\pi}{4}.
\end{aligned}$$

Also solved by Brian Bradie, Kenneth B. Davenport, Russell Jay Hendel, Harris Kwong, Ángel Plaza, Dan Weiner, part (i) only, and the proposer.

Based on Binomial Coefficients and Cauchy-Schwartz

B-1162 **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 53.1, February 2015)**

Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \sqrt{\binom{n-1}{k-1}\frac{F_k}{k}} \le \sqrt{F_{2n}}.$$

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

First note that

$$\frac{1}{k}\binom{n-1}{k-1} = \frac{1}{n}\binom{n}{k},$$

so that

$$\sum_{k=1}^{n} \sqrt{\binom{n-1}{k-1}\frac{F_k}{k}} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\sqrt{\binom{n}{k}F_k}. \tag{1}$$

Now, by Jensen's inequality,

$$\sum_{k=1}^{n}\sqrt{\binom{n}{k}F_k} \le n\cdot\sqrt{\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}F_k} = \sqrt{n}\cdot\sqrt{\sum_{k=1}^{n}\binom{n}{k}F_k}. \tag{2}$$

Moreover,

$$\sum_{k=1}^{n}\binom{n}{k}F_k = F_{2n} \tag{3}$$

(see Identity 6 on page 6 of [1]). Finally, combining (1), (2), and (3) yields

$$\sum_{k=1}^{n} \sqrt{\binom{n-1}{k-1}\frac{F_k}{k}} \le \sqrt{F_{2n}}.$$

References

[1] A. Benjamin and J. Quinn, *Proofs that Really Count*, The Mathematical Association of America, Washington, D. C., 2003.

Also solved by Kenneth B. Davenport, Russell Jay Hendel, Zbigniew Jakubczyk, Harris Kwong, Hideyuki Ohtsuka, Ángel Plaza, Nicuşor Zlota, and the proposer.

Two Partial Sums Involving k-Fibonacci and k-Lucas Sequences and One Lower Bound

B-1163 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 53.1, February 2015)

For any positive integer k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$ with respective initial values $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. For any integer $n \geq 2$, prove that

(i) $$\sum_{j=1}^{n}\left(\frac{kF_{k,j}}{F_{k,n+1}+F_{k,n}-1-kF_{k,j}}\right)^2 \geq \frac{n}{(n-1)^2},$$

(ii) $$\sum_{j=1}^{n}\left(\frac{kL_{k,j}}{L_{k,n+1}+L_{k,n}-2-k-kL_{k,j}}\right)^2 \geq \frac{n}{(n-1)^2}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

It can be proved by induction that $\sum_{j=1}^{n} kF_{k,j} = F_{k,n+1} + F_{k,n} - 1$, and $\sum_{j=1}^{n} kL_k, j = L_{k,n+1} + L_{k,n} - 2 - k$. Thus, both (i) and (ii) take the form of

$$\sum_{j=1}^{n}\left(\frac{x_j}{S-x_j}\right)^2 \geq \frac{n}{(n-1)^2},$$

where $\{x_j\}_{j=1}^{n}$ is a non-decreasing sequence of positive integers, and $S = \sum_{j=1}^{n} x_j$. Taking note that the sequence $\{\frac{1}{S-x_j}\}_{j=1}^{n}$ is also non-decreasing, we deduce from Chebyshev and Cauchy-Schwartz inequalities that

$$\sum_{j=1}^{n}\left(\frac{x_j}{S-x_j}\right)^2 \geq \frac{1}{n}\left(\sum_{j=1}^{n} x_j^2\right)\left(\sum_{j=1}^{n}\frac{1}{(S-x_j)^2}\right) \geq \frac{1}{n^3}\left(\sum_{j=1}^{n} x_j\right)^2\left(\sum_{j=1}^{n}\frac{1}{S-x_j}\right)^2.$$

It is obvious that $(\sum_{j=1}^{n} x_j)^2 = S^2$. Since the harmonic mean of a finite number of positive real numbers is less than or equal to their arithmetic mean, we find

$$\frac{1}{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{S-x_j}} \leq \frac{1}{n}\sum_{j=1}^{n}(S-x_j) = \frac{(n-1)S}{n}.$$

Hence,

$$\sum_{j=1}^{n}\frac{1}{S-x_j} \geq \frac{n^2}{(n-1)S},$$

from which the desired result follows.

Also solved by D. M. Bătineţu–Giurgiu and Neculai Stanciu (jointly), and the proposer.

The Values of Two Alternating Series

B-1164 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 53.1, February 2015)

Determine each of the following:

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2F_n} L_{2F_{n+3}}}$,

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2L_n} L_{2L_{n+3}}}$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

Using the identity $L_s L_t = L_{s+t} + (-1)^s L_{t-s}$, we find

$$L_{2F_{n+1}} L_{2F_{n+2}} = L_{2F_{n+3}} + L_{2F_n},$$

from which we deduce the identity

$$\frac{1}{L_{2F_n} L_{2F_{n+3}}} = \frac{L_{2F_{n+3}} + L_{2F_n}}{L_{2F_n} L_{2F_{n+1}} L_{2F_{n+2}} L_{2F_{n+3}}} = \frac{1}{L_{2F_n} L_{2F_{n+1}} L_{2F_{n+2}}} + \frac{1}{L_{2F_{n+1}} L_{2F_{n+2}} L_{2F_{n+3}}}.$$

Its alternating sum is telescopic. Consequently,

$$\sum_{n=0}^{m} \frac{(-1)^n}{L_{2F_n} L_{2F_{n+3}}} = \frac{1}{L_{2F_0} L_{2F_1} L_{2F_2}} + \frac{(-1)^m}{L_{2F_{m+1}} L_{2F_{m+2}} L_{2F_{m+3}}},$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2F_n} L_{2F_{n+3}}} = \lim_{m\to\infty} \left(\frac{1}{L_{2F_0} L_{2F_1} L_{2F_2}} + \frac{(-1)^m}{L_{2F_{m+1}} L_{2F_{m+2}} L_{2F_{m+3}}} \right) = \frac{1}{L_{2F_0} L_{2F_1} L_{2F_2}} = \frac{1}{18}.$$

In a similar manner, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2L_n} L_{2L_{n+3}}} = \lim_{m\to\infty} \left(\frac{1}{L_{2L_0} L_{2L_1} L_{2L_2}} + \frac{(-1)^m}{L_{2L_{m+1}} L_{2L_{m+2}} L_{2L_{m+3}}} \right) = \frac{1}{L_{2L_0} L_{2L_1} L_{2L_2}} = \frac{1}{378}.$$

Also solved by Brian Bradie, Kenneth B. Davenport, and the proposer.

Fibonacci Indices

B-1165 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 53.1, February 2015)

For an integer $n \geq 0$, find the value of

$$\frac{L_{F_{3n}}}{F_{F_{3n-1}} F_{F_{3n-2}}} + \frac{L_{F_{3n-1}}}{F_{F_{3n-2}} F_{F_{3n}}} + \frac{L_{F_{3n-2}}}{F_{F_{3n}} F_{F_{3n-1}}}.$$

Solution by John M. Adams, Charleston, SC.

The original expression is equivalent to

$$\frac{L_{F_{3n}}F_{F_{3n}} + L_{F_{3n-1}}F_{F_{3n-1}} + L_{F_{3n-2}}F_{F_{3n-2}}}{F_{F_{3n}}F_{F_{3n-1}}F_{F_{3n-2}}}. \tag{1}$$

Using the Lucas formula and Binet's formula for the numerator, we can see that

$$L_{F_{3n}}F_{F_{3n}} = \left(\alpha^{F_{3n}} + \beta^{F_{3n}}\right)\left(\frac{\alpha^{F_{3n}} - \beta^{F_{3n}}}{\sqrt{5}}\right),$$

$$L_{F_{3n-1}}F_{F_{3n-1}} = \left(\alpha^{F_{3n-1}} + \beta^{F_{3n-1}}\right)\left(\frac{\alpha^{F_{3n-1}} - \beta^{F_{3n-1}}}{\sqrt{5}}\right),$$

$$L_{F_{3n-2}}F_{F_{3n-2}} = \left(\alpha^{F_{3n-2}} + \beta^{F_{3n-2}}\right)\left(\frac{\alpha^{F_{3n-2}} - \beta^{F_{3n-2}}}{\sqrt{5}}\right).$$

Multiplying the expressions and using the distributive property, we obtain

$$\left(\frac{1}{\sqrt{5}}\right)\left(\alpha^{2F_{3n}} - \beta^{2F_{3n}} + \alpha^{2F_{3n-1}} - \beta^{2F_{3n-1}} + \alpha^{2F_{3n-2}} - \beta^{2F_{3n-2}}\right). \tag{2}$$

Using Binet's formula for the numerator, we see that

$$F_{F_{3n}}F_{F_{3n-1}}F_{F_{3n-2}} = \left(\frac{\alpha^{F_{3n}} - \beta^{F_{3n}}}{\sqrt{5}}\right)\left(\frac{\alpha^{F_{3n-1}} - \beta^{F_{3n-1}}}{\sqrt{5}}\right)\left(\frac{\alpha^{F_{3n-2}} - \beta^{F_{3n-2}}}{\sqrt{5}}\right).$$

Multiplying, simplifying, reordering, and factoring, we obtain

$$\left(\frac{1}{5\sqrt{5}}\right)\left(\alpha^{2F_{3n}} - \beta^{2F_{3n}} + (\alpha\beta)^{F_{3n-2}}(-\alpha^{2F_{3n-1}} + \beta^{2F_{3n-1}}) + (\alpha\beta)^{F_{3n-1}}(-\alpha^{2F_{3n-2}} + \beta^{2F_{3n-2}})\right).$$

Since $\alpha\beta = -1$ and F_{3n-2} and F_{3n-1} are both odd for any integer n, $(\alpha\beta)^{F_{3n-1}} = (\alpha\beta)^{F_{3n-2}} = -1$. Thus, the denominator is equivalent to

$$\left(\frac{1}{5\sqrt{5}}\right)\left(\alpha^{2F_{3n}} - \beta^{2F_{3n}} + \alpha^{2F_{3n-1}} - \beta^{2F_{3n-1}} + \alpha^{2F_{3n-2}} - \beta^{2F_{3n-2}}\right). \tag{3}$$

Replacing (2) and (3) as the numerator and denominator in (1), we have

$$\frac{\left(\frac{1}{\sqrt{5}}\right)\left(\alpha^{2F_{3n}} - \beta^{2F_{3n}} + \alpha^{2F_{3n-1}} - \beta^{2F_{3n-1}} + \alpha^{2F_{3n-2}} - \beta^{2F_{3n-2}}\right)}{\left(\frac{1}{5\sqrt{5}}\right)\left(\alpha^{2F_{3n}} - \beta^{2F_{3n}} + \alpha^{2F_{3n-1}} - \beta^{2F_{3n-1}} + \alpha^{2F_{3n-2}} - \beta^{2F_{3n-2}}\right)} = 5.$$

Therefore,

$$\frac{L_{F_{3n}}}{F_{F_{3n-1}}F_{F_{3n-2}}} + \frac{L_{F_{3n-1}}}{F_{F_{3n-2}}F_{F_{3n}}} + \frac{L_{F_{3n-2}}}{F_{F_{3n}}F_{F_{3n-1}}} = 5.$$

Also solved by Brian Bradie, Kenneth B. Davenport, Harris Kwong, Wei-Kai Lai, Ángel Plaza and Francisco Perdomo (jointly), and the proposer.

The solver to Problem B-1156 was Albert Stadler. We apologize for the inadvertant misspelling of his name.