ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
HARRIS KWONG

Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2017. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

Dedication: The problems in this issue are dedicated to Dr. Russ Euler and Dr. Jawad Sadek in recognition of their sixteen years of devoted service as the editors of the Elementary Problems and Solutions Section.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1201 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

If $a, b, c > 0$, then prove that, for any positive integer n,

$$\frac{a^3}{aF_n + bF_{n+1}} + \frac{b^3}{bF_n + aF_{n+1}} \geq \frac{a^2+b^2}{F_{n+2}},$$

$$\frac{a^3}{aL_n + bL_{n+1}} + \frac{b^3}{bL_n + aL_{n+1}} \geq \frac{a^2+b^2}{L_{n+2}},$$

$$\frac{a^3}{aF_n + bF_{n+1} + cF_{n+2}} + \frac{b^3}{bF_n + cF_{n+1} + aF_{n+2}} + \frac{c^3}{cF_n + aF_{n+1} + bF_{n+2}} \geq \frac{a^2+b^2+c^2}{2F_{n+2}},$$

$$\frac{a^3}{aL_n + bL_{n+1} + cL_{n+2}} + \frac{b^3}{bL_n + cL_{n+1} + aL_{n+2}} + \frac{c^3}{cL_n + aL_{n+1} + bL_{n+2}} \geq \frac{a^2+b^2+c^2}{2L_{n+2}}.$$

B-1202 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, Neculai Stanciu, George Emil Palade School, Buzău, Romaina, and Gabriel Tica, Mihai Viteazul National College, Băileşti, Dolj, Romania.

Let $(a_n)_{n\geq 1}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a$. Evaluate

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_nF_n}{n!}}\right) \quad \text{and} \quad \lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_nL_n}{n!}}\right).$$

B-1203 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that, for any positive integer n,

(i) $\displaystyle\sum_{k=1}^{n} F_{F_{3k}}F_{F_{3k-1}}F_{F_{3k-2}} = \frac{1}{5}\sum_{k=1}^{3n} F_{2F_k}$;

(ii) $\displaystyle\sum_{k=1}^{n} L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} = 2n + \sum_{k=1}^{3n} (-1)^{L_k} L_{2L_k}$.

B-1204 Proposed by Steve Edwards, Kennesaw State University, Marietta, GA.

For non-negative integers n, express

$$A_n = \sum_{j=0}^{n} \frac{1}{2^{2j}} \sum_{i=0}^{n+j} \binom{n+j-i}{n-j}\binom{n+j}{i} \quad \text{and} \quad B_n = \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \sum_{i=0}^{n+j} \binom{n+j-i}{n-j-1}\binom{n+j}{i}$$

in terms of Fibonacci numbers.

B-1205 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that

$$n^{m-1}\sum_{k=1}^{n} F_k^{2m} \geq F_n^m F_{n+1}^m$$

for any positive integers n and m.

SOLUTIONS

Adding Two Terms at a Time

B-1181 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 54.1, February 2016)

Prove that

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{(F_n^2+1)^2} = \frac{5}{3}.$$

Solution by Jeremiah Bartz, University of North Dakota, Grand Forks, ND.

We make use of the following four identities:

(1) $3F_{2i} = F_{i+2}^2 - F_{i-2}^2$.
(2) $F_{2i+1}^2 + 1 = F_{2i-1}F_{2i+3}$.
(3) $F_{2i} = F_{i+1}^2 - F_{i-1}^2$. (See [1, p. 79]).
(4) For even i, $F_i^2 + 1 = F_{i-1}F_{i+1}$. (Special case of Cassini's Formula [1, p. 74]).

The first identity can be derived as follows. From repeated application of the relation $F_i = F_{i+1} - F_{i-1}$, it follows that $F_i = F_{i+2} - 2F_i + F_{i-2}$, or equivalently, $3F_i = F_{i+2} + F_{i-2}$. Then using the identities $F_{2i} = F_iL_i$ and $L_i = F_{i+2} - F_{i-2}$ (see [1, p. 80]), we see that

$$3F_{2i} = 3F_iL_i = (F_{i+2} + F_{i-2})(F_{i+2} - F_{i-2}) = F_{i+2}^2 - F_{i-2}^2.$$

Identity (2) is a special case of a result due to Sharpe (see Identity 54 of [1, p. 90]). Using (1)–(4), the given series can be expressed as a telescoping series. For $M \geq 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{F_{2n}}{(F_n^2+1)^2} &= \frac{F_2}{(F_1^2+1)^2} + \frac{F_4}{(F_2^2+1)^2} + \lim_{N\to\infty} \sum_{n=3}^{N} \frac{F_{2n}}{(F_n^2+1)^2} \\
&= \frac{1}{4} + \frac{3}{4} + \lim_{M\to\infty} \sum_{k=1}^{M} \left(\frac{3F_{2(2k+1)}}{3(F_{2k+1}^2+1)^2} + \frac{F_{2(2k+2)}}{(F_{2k+2}^2+1)^2} \right) \\
&= 1 + \lim_{M\to\infty} \sum_{k=1}^{M} \left(\frac{F_{2k+3}^2 - F_{2k-1}^2}{3(F_{2k-1}F_{2k+3})^2} + \frac{F_{2k+3}^2 - F_{2k+1}^2}{(F_{2k+1}F_{2k+3})^2} \right) \\
&= 1 + \lim_{M\to\infty} \sum_{k=1}^{M} \left(\frac{1}{3F_{2k-1}^2} - \frac{1}{3F_{2k+3}^2} + \frac{1}{F_{2k+1}^2} - \frac{1}{F_{2k+3}^2} \right) \\
&= 1 + \lim_{M\to\infty} \left(\frac{1}{3F_1^2} + \frac{1}{3F_3^2} + \frac{1}{F_3^2} - \frac{1}{3F_{2M+1}^2} - \frac{1}{3F_{2M+3}^2} - \frac{1}{F_{2M+3}^2} \right).
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{(F_n^2+1)^2} = 1 + \frac{1}{3F_1^2} + \frac{1}{3F_3^2} + \frac{1}{F_3^2} = 1 + \frac{1}{3} + \frac{1}{12} + \frac{1}{4} = \frac{5}{3}.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, NY, 2001.

Also solved by Brian Bradie, Steve Edwards, Harris Kwong, Poo-Sung Park, Jason L. Smith, and the proposer.

Vieta's Formula

B-1182 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 54.1, February 2016)

Let n be a positive integer. If a, b, c are the roots of the equation $x^3 - F_nx^2 + F_{n+1} = 0$, then prove that

$$a^3(F_n - a) + b^3(F_n - b) + c^3(F_n - c)$$

is a positive integer which is the sum of n squares.

Solution by Hideyuki Ohtsuka, Saitama, Japan.

For any root r of the equation $x^3 - F_n x^2 + F_{n+1} = 0$ we have

$$r^3(F_n - r) = r(r^2 F_n - r^3) = rF_{n+1}.$$

In addition, Vieta's formula asserts that $a + b + c = F_n$. Using the above identity, we find

$$a^3(F_n - a) + b^3(F_n - b) + c^3(F_n - c) = (a + b + c)F_{n+1} = F_n F_{n+1} = \sum_{k=1}^{n} F_k^2.$$

Thus, the given sum is a positive integer which is the sum of n squares.

Editor's Remark: A similar result holds when the Fibonacci numbers in the problem are replace by the Lucas numbers. The sum equals to $L_n L_{n+1} = 2 + \sum_{k=1}^{n} L_k^2$.

Also solved by Adnan Ali (student), Brian D. Beasley, Charles K. Cook, Itzal De Urioste (student), Steve Edwards, I. V. Fedak, Dmitry Fleischman, G. C. Gruebel, Russell Jay Hendel, Harris Kwong, Ángel Plaza, Vincelot Ravoson, Jason L. Smith, David Terr, and the proposer.

A Binomial Series

B-1183 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 54.1, February 2016)

Express

$$\sum_{n=k}^{\infty} \binom{n}{k} \frac{2}{L_{n+1} + \sqrt{5}\, F_{n+1}}$$

as a function of F_{k+1} and L_{k+1}.

Solution by Adnan Ali (student), A.E.C.S-4, Mumbai, India.

First, we note that

$$L_{n+1} + \sqrt{5}\, F_{n+1} = 2\alpha^{n+1},$$

and so the sum to be evaluated becomes

$$\sum_{n=k}^{\infty} \binom{n}{k} \frac{2}{L_{n+1} + \sqrt{5}\, F_{n+1}} = \sum_{n=k}^{\infty} \binom{n}{k} (\alpha^{-1})^{n+1}.$$

We know that

$$\sum_{n=k}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1 - x)^{k+1}},$$

and so

$$\sum_{n=k}^{\infty} \binom{n}{k} (\alpha^{-1})^{n+1} = \left(\frac{\alpha^{-1}}{1 - \alpha^{-1}} \right)^{k+1} = \alpha^{k+1} = \frac{L_{k+1} + \sqrt{5}\, F_{k+1}}{2}.$$

Also solved by Brian Bradie, Steve Edwards, G. C. Greubel, Harris Kwong, Hideyuki Ohtsuka, S. S. Pradhan and M. K. Sahukar, Jason L. Smith, David Terr, and the proposer.

Matrices of Ones

B-1184 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romaina.
(Vol. 54.1, February 2016)

Let k be a positive integer.

(1) If $A(k) = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$, compute $\prod_{k=1}^{n} A(k)$.

(2) If $B(k) = \begin{pmatrix} F_k^2 & F_{k+1}^2 \\ F_{k+1}^2 & F_k^2 \end{pmatrix}$, compute $\prod_{k=1}^{n} B(k)$.

(3) If $C(k) = \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \begin{pmatrix} L_k & L_{k+1} \\ L_{k+1} & L_k \end{pmatrix}$, compute $\prod_{k=1}^{n} B(k)$.

Solution by Vincelot Ravoson, Paris, France.

Let $J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Notice that $J \begin{pmatrix} a & b \\ b & a \end{pmatrix} = (a+b)J$ for any real numbers a and b.

(1) Let us prove by induction that

$$\prod_{k=1}^{n} A(k) = \left(\frac{1}{2} \prod_{k=1}^{n+2} F_k \right) J.$$

For $n = 1$, we have $A(1) = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_2 \end{pmatrix} = J = \frac{1}{2} F_1 F_2 F_3 J$. Now, we assume that the result is true for a fixed positive integer n. Then

$$\prod_{k=1}^{n+1} A(k) = \left(\prod_{k=1}^{n} A(k) \right) A(n+1) = \left(\frac{1}{2} \prod_{k=1}^{n+2} F_k \right) JA(n+1).$$

Using the remark above,

$$JA(n+1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix} = (F_{n+1} + F_{n+2})J = F_{n+3}J.$$

Thus,

$$\prod_{k=1}^{n+1} A(k) = \left(\frac{1}{2} \prod_{k=1}^{n+2} F_k \right) \cdot F_{n+3} J = \left(\frac{1}{2} \prod_{k=1}^{n+3} F_k \right) J,$$

and the conclusion follows.

(2) Let us prove by induction that

$$\prod_{k=1}^{n} B(k) = \left(\frac{1}{2} \prod_{k=1}^{n} F_{2k+1} \right) J.$$

The proof is essentially the same as the proof of (1), except that, in the inductive step, we use the identity $F_{n+1}^2 + F_{n+2}^2 = F_{2n+3}$ to prove that

$$JB(n+1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n+1}^2 & F_{n+2}^2 \\ F_{n+2}^2 & F_{n+1}^2 \end{pmatrix} = (F_{n+1}^2 + F_{n+2}^2)J = F_{2n+3}J.$$

(3) Let us prove by induction that

$$\prod_{k=1}^{n} C(k) = \left(\frac{1}{6} \prod_{k=1}^{n+2} F_{2k} \right) J.$$

The proof is almost identical to those of (1) and (2), except that, in the inductive step,

$$\begin{aligned} JC(n+1) &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_{n+2} \\ F_{n+2} & F_{n+1} \end{pmatrix} \begin{pmatrix} L_{n+1} & L_{n+2} \\ L_{n+2} & L_{n+1} \end{pmatrix} \\ &= (F_{n+1} + F_{n+2})J \begin{pmatrix} L_{n+1} & L_{n+2} \\ L_{n+2} & L_{n+1} \end{pmatrix} \\ &= (F_{n+1} + F_{n+2})(L_{n+1} + L_{n+2})J \\ &= F_{n+3}L_{n+3}J \\ &= F_{2n+6}J. \end{aligned}$$

Also solved by Jeremiah Bartz, Brian Bradie, Steve Edwards, Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Jason L. Smith, David Stone and John Hawkins, David Terr, and the proposer.

Chebyshev, Jensen, or Nesbitt?

B-1185 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.
(Vol. 54.1, February 2016)

Let a and b be positive real numbers such that $aF_{n+1} > bF_n$. Prove that

$$\sum_{k=1}^{n} \frac{F_k^2}{aF_nF_{n+1} - bF_k^2} \geq \frac{n}{an-b}$$

for all $n \geq 1$.

Solution 1 by Hideyuki Ohtsuka, Saitama, Japan.

Note that for $1 \leq k \leq n$,

$$aF_nF_{n+1} - bF_k^2 \geq aF_nF_{n+1} - bF_n^2 = F_n(aF_{n+1} - bF_n) > 0,$$

and for $1 \leq k \leq n-1$, we have

$$F_k^2 \leq F_{k+1}^2 \qquad \text{and} \qquad \frac{1}{aF_nF_{n+1} - bF_k^2} \leq \frac{1}{aF_nF_{n+1} - bF_{k+1}^2}.$$

By Chebyshev's inequality and the AM-HM inequality,

$$\sum_{k=1}^{n} \frac{F_k^2}{aF_nF_{n+1} - bF_k^2} \geq \frac{1}{n} \left(\sum_{k=1}^{n} F_k^2 \right) \left(\sum_{k=1}^{n} \frac{1}{aF_nF_{n+1} - bF_k^2} \right) \geq \frac{n \sum_{k=1}^{n} F_k^2}{\sum_{k=1}^{n} (aF_nF_{n+1} - bF_k^2)}.$$

Note that $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$, we have

$$\sum_{k=1}^{n} \frac{F_k^2}{aF_nF_{n+1} - bF_k^2} \geq \frac{nF_nF_{n+1}}{anF_nF_{n+1} - bF_nF_{n+1}} = \frac{n}{an-b}.$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA.

Given $aF_{n+1} > bF_n$, it follows that $aF_nF_{n+1} > bF_n^2 \geq bF_k^2$. Let $f(x) = x/(aF_nF_{n+1} - bx)$. Then

$$f''(x) = \frac{2abF_nF_{n+1}}{(aF_nF_{n+1} - bx)^2} > 0$$

for $x \in \left[F_1^2, F_n^2\right]$, so f is convex in this interval. Now, Jensen's inequality yields

$$\sum_{k=1}^{n} \frac{F_k^2}{aF_nF_{n+1} - bF_k^2} = \sum_{k=1}^{n} f(F_k^2) \geq n \cdot f\left(\frac{1}{n}\sum_{k=1}^{n} F_k^2\right).$$

Since $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$,

$$\sum_{k=1}^{n} \frac{F_k^2}{aF_nF_{n+1} - bF_k^2} \geq n \cdot \frac{\frac{1}{n}F_nF_{n+1}}{aF_nF_{n+1} - b \cdot \frac{1}{n}F_nF_{n+1}} = \frac{n}{an-b}.$$

Solution 3 by G. C. Greubel, Newport News, VA.

By making use of the generalized Nesbitt's inequality (see, for example, [1])

$$\sum_{k=1}^{n} \frac{x_k^{\alpha+1}}{\left(a + \frac{c}{n}\right)\left(\sum_{j=1}^{n} x_j\right) - bx_k} \geq \frac{n^{1-\alpha}}{an+c-b}\left(\sum_{k=1}^{n} x_k\right)^{\alpha}$$

for the case of $x_k = F_k^2$ and $c = 0$, the given inequality becomes

$$\sum_{k=1}^{n} \frac{F_k^{2\alpha+2}}{aF_nF_{n+1} - bF_k^2} \geq \frac{n^{1-\alpha}(F_nF_{n+1})^{\alpha}}{an-b}.$$

When $\alpha = 0$ the desired result is obtained, namely,

$$\sum_{k=1}^{n} \frac{F_k^2}{aF_nF_{n+1} - bF_k^2} \geq \frac{n}{an-b}.$$

When $\alpha = 1$ the inequality becomes

$$\sum_{k=1}^{n} \frac{F_k^4}{aF_nF_{n+1} - bF_k^2} \geq \frac{F_nF_{n+1}}{an-b}.$$

References

[1] M. Bencze and O. T. Pop, *Generalizations and refinements for Nesbitt's inequality*, J. Math. Inequal., **5** (2011), 13–20.

Also solved by Adnan Ali (student), Ángel Plaza, and the proposer.