ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2018. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1221 Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (Barcelona Tech), Barcelona, Spain.

For any positive integer n, show that

$$\frac{1}{54F_{2n}} \begin{vmatrix} 4 & F_n & L_n \\ F_n & (F_{n+1}+L_n)^2 & F_{2n} \\ L_n & F_{2n} & F_{n+2}^2 \end{vmatrix}$$

is a perfect square, and find its value.

B-1222 Proposed by Kenny B. Davenport, Dallas, PA.

Let H_n denote the nth harmonic number. Prove that

$$\sum_{n=2}^{\infty} \frac{H_{n-1}F_n}{n2^n} = \frac{\ln 16 \cdot \ln \alpha}{\sqrt{5}}, \qquad \text{and} \qquad \sum_{n=2}^{\infty} \frac{H_{n-1}L_n}{n2^n} = (\ln 2)^2 + 4(\ln \alpha)^2.$$

B-1223 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n and a, prove that

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a - F_{n+2}^a - 1) \leq 0.$$

B-1224 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any positive integer n, prove that

$$\sum_{k=1}^{n} \binom{n}{k} \frac{F_k}{k} = \sum_{k=1}^{n} \frac{F_{2k}}{k}, \qquad \text{and} \qquad \sum_{k=1}^{n} \binom{n}{k} \frac{L_k}{k} = \sum_{k=1}^{n} \frac{L_{2k} - 2}{k}.$$

B-1225 Proposed by Jathan Austin, Salisbury University, Salisbury, MD.

Construct a sequence $\{M_n\}_{n=1}^{\infty}$ of 3×3 matrices with positive entries that satisfy the following conditions:

(A) $|M_n|$ is the product of nonzero Fibonacci numbers.
(B) The determinant of any 2×2 submatrix of M_n is a Fibonacci number or the product of nonzero Fibonacci numbers.
(C) $\lim_{n\to\infty} |M_{n+1}|/|M_n| = 1 + 2\alpha$.

SOLUTIONS

Cauchy-Schwarz or Bergström Again!

B-1201 Proposed by Ivan V. Fedak, Vasyl Stefanyc Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 55.1, February 2017)

If $a, b, c > 0$, then prove that, for any positive integer n,

$$\frac{a^3}{aF_n + bF_{n+1}} + \frac{b^3}{bF_n + aF_{n+1}} \geq \frac{a^2 + b^2}{F_{n+2}},$$
$$\frac{a^3}{aL_n + bL_{n+1}} + \frac{b^3}{bL_n + aL_{n+1}} \geq \frac{a^2 + b^2}{L_{n+2}},$$
$$\frac{a^3}{aF_n + bF_{n+1} + cF_{n+2}} + \frac{b^3}{bF_n + cF_{n+1} + aF_{n+2}} + \frac{c^3}{cF_n + aF_{n+1} + bF_{n+2}} \geq \frac{a^2 + b^2 + c^2}{2F_{n+2}},$$
$$\frac{a^3}{aL_n + bL_{n+1} + cL_{n+2}} + \frac{b^3}{bL_n + cL_{n+1} + aL_{n+2}} + \frac{c^3}{cL_n + aL_{n+1} + bL_{n+2}} \geq \frac{a^2 + b^2 + c^2}{2L_{n+2}}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Let a, b, c, x, y, z be positive real numbers. By the Cauchy-Schwarz inequality,

$$\frac{a^3}{ax + by} + \frac{b^3}{bx + cz} = \frac{a^4}{a^2x + aby} + \frac{b^4}{b^2x + aby} \geq \frac{(a^2 + b^2)^2}{(a^2 + b^2)x + 2aby}.$$

Now, by the arithmetic mean - geometric mean inequality, $2ab \leq a^2 + b^2$, so

$$\frac{a^3}{ax+by} + \frac{b^3}{bx+cz} \geq \frac{(a^2+b^2)^2}{(a^2+b^2)x+(a^2+b^2)y} = \frac{a^2+b^2}{x+y}. \tag{1}$$

With $x = F_n$ and $y = F_{n+1}$, (1) becomes

$$\frac{a^3}{aF_n+bF_{n+1}} + \frac{b^3}{bF_n+aF_{n+1}} \geq \frac{a^2+b^2}{F_n+F_{n+1}} = \frac{a^2+b^2}{F_{n+2}}.$$

With $x = L_n$ and $y = L_{n+1}$, (1) becomes

$$\frac{a^3}{aL_n+bL_{n+1}} + \frac{b^3}{bL_n+aL_{n+1}} \geq \frac{a^2+b^2}{L_n+L_{n+1}} = \frac{a^2+b^2}{L_{n+2}}.$$

Next, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\frac{a^3}{ax+by+cz} + \frac{b^3}{bx+cy+az} + \frac{c^3}{cx+ay+bz} \\ &= \frac{a^4}{a^2x+aby+caz} + \frac{b^4}{b^2x+bcy+abz} + \frac{c^4}{c^2x+cay+bcz} \\ &\geq \frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)x+(ab+bc+ca)(y+z)}. \end{aligned}$$

Now, the inequality $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$ is equivalent to

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

so

$$\begin{aligned} &\frac{a^3}{ax+by+cz} + \frac{b^3}{bx+cy+az} + \frac{c^3}{cx+ay+bz} \\ &\geq \frac{(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)x+(a^2+b^2+c^2)(y+z)} = \frac{a^2+b^2+c^2}{x+y+z}. \end{aligned} \tag{2}$$

With $x = F_n$, $y = F_{n+1}$, and $z = F_{n+2}$, (2) becomes

$$\begin{aligned} &\frac{a^3}{aF_n+bF_{n+1}+cF_{n+2}} + \frac{b^3}{bF_n+cF_{n+1}+aF_{n+2}} + \frac{c^3}{cF_n+aF_{n+1}+bF_{n+2}} \\ &\geq \frac{a^2+b^2+c^2}{F_n+F_{n+1}+F_{n+2}} = \frac{a^2+b^2+c^2}{2F_{n+2}}. \end{aligned}$$

With $x = L_n$, $y = L_{n+1}$, and $z = L_{n+2}$, (2) becomes

$$\begin{aligned} &\frac{a^3}{aL_n+bL_{n+1}+cL_{n+2}} + \frac{b^3}{bL_n+cL_{n+1}+aL_{n+2}} + \frac{c^3}{cL_n+aL_{n+1}+bL_{n+2}} \\ &\geq \frac{a^2+b^2+c^2}{L_n+L_{n+1}+L_{n+2}} = \frac{a^2+b^2+c^2}{2L_{n+2}}. \end{aligned}$$

Editor's Note: Ricardo used Bergström inequality to derive (1) and (2).

Also solved by Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Henry Ricardo, Nicuşor Zlota, and the proposer.

Root and Ratio Tests

B-1202 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania; Neculai Stanciu, George Emil Palade School, Buzău, Romaina; and Gabriel Tica, Mihai Viteazul National College, Băileşti, Dolj, Romania.
(Vol. 55.1, February 2017)

Let $(a_n)_{n\geq 1}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a$. Evaluate

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_nF_n}{n!}}\right) \quad \text{and} \quad \lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_nL_n}{n!}}\right).$$

Solution by the proposers.

We claim that both limits equal to $a\alpha/e$. Given an infinite sequence $(b_n)_{n\geq 1}$, it is known that if $\lim_{n\to\infty} |b_{n+1}/b_n| = L$, then $\lim_{n\to\infty} \sqrt[n]{|b_n|} = L$. Apply this to $c_n = a_nF_n/(n!\,n^n)$. We find

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{a_{n+1}F_{n+1}}{(n+1)!\,(n+1)^{n+1}}\cdot\frac{n!\,n^n}{a_nF_n} = \lim_{n\to\infty}\frac{a_{n+1}}{n^2a_n}\cdot\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^{n+2} = \frac{a\alpha}{e}.$$

Thus, $\lim_{n\to\infty}\sqrt[n]{c_n} = a\alpha/e$ as well. Define

$$u_n = \sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(n+1)!}}\cdot\sqrt[n]{\frac{n!}{a_nF_n}} = \frac{\sqrt[n+1]{c_{n+1}}}{\sqrt[n]{c_n}}\cdot\frac{n+1}{n},$$

such that

$$\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_nF_n}{n!}} = \sqrt[n]{\frac{a_nF_n}{n!}}\,\bigl(u_n - 1\bigr) = \sqrt[n]{c_n}\cdot n(u_n-1).$$

It suffices to show that $\lim_{n\to\infty} n(u_n-1) = 1$. Note that $\lim_{n\to\infty} u_n = 1$, and

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{c_{n+1}}{c_n}\cdot\frac{1}{\sqrt[n+1]{c_{n+1}}}\left(\frac{n+1}{n}\right)^n = e.$$

Therefore,

$$\lim_{n\to\infty} n(u_n-1) = \lim_{n\to\infty}\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n = 1\cdot\ln e = 1.$$

The proof of the other limit is similar, and is omitted here.

Editor's Note: Plaza noted that the inequalities follow from a result obtained by the first two proposers in [1], and Ohtsuka used a result from [2] to derive the inequalities directly.

References

[1] D. M. Bătineţu-Giurgiu and N. Stanciu, *New methods for calculations of some limits*, The Teaching of Mathematics, **16**(2) (2013), 82–88.
[2] Gh. Toader, *Lalescu sequences*, Publikacije Elektrotechničkog fakulteta Univerziteta u Beogradu, Serija Matematika i fizika, **9** (1998), 19–28.

Also solved by I. V. Fedak, Dmitry Fleishcman, Hamza Mahmood (student), Soumitra Mandal, Hideyuki Ohtsuka, Ángel Plaza, and Raphael Schumacher (student).

Fibonacci Numbers with Fibonacci Numbers as Subscripts

B-1203 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 55.1, February 2017)

Prove that, for any positive integer n,

(i) $\displaystyle\sum_{k=1}^{n} F_{F_{3k}}F_{F_{3k-1}}F_{F_{3k-2}} = \frac{1}{5}\sum_{k=1}^{3n} F_{2F_k}$;

(ii) $\displaystyle\sum_{k=1}^{n} L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} = 2n + \sum_{k=1}^{3n}(-1)^{L_k}L_{2L_k}$.

Solution by Jaroslav Seibert, University of Pardubice, Czech Republic.

Using the Binet's formula for the Fibonacci numbers, we find

$$\begin{aligned}F_{F_{3k}}F_{F_{3k-1}}F_{F_{3k-2}} &= \left(\frac{\alpha^{F_{3k}}-\beta^{F_{3k}}}{\sqrt{5}}\right)\left(\frac{\alpha^{F_{3k-1}}-\beta^{F_{3k-1}}}{\sqrt{5}}\right)\left(\frac{\alpha^{F_{3k-2}}-\beta^{F_{3k-2}}}{\sqrt{5}}\right)\\ &= \frac{1}{5\sqrt{5}}\left[\alpha^{2F_{3k}} - (\alpha\beta)^{F_{3k-2}}\alpha^{2F_{3k-1}} - (\alpha\beta)^{F_{3k-1}}\alpha^{2F_{3k-2}} - (\alpha\beta)^{F_{3k}}\right.\\ &\quad\left. + (\alpha\beta)^{F_{3k}} + (\alpha\beta)^{F_{3k-1}}\beta^{2F_{3k-2}} + (\alpha\beta)^{F_{3k-2}}\beta^{2F_{3k-1}} - \beta^{2F_{3k}}\right].\end{aligned}$$

Since $\alpha\beta = -1$, and F_{3k-1} and F_{3k-2} are both odd for any integer k, we have $(\alpha\beta)^{F_{3k-1}} = (\alpha\beta)^{F_{3k-2}} = -1$. Thus,

$$\begin{aligned}F_{F_{3k}}F_{F_{3k-1}}F_{F_{3k-2}} &= \frac{1}{5}\left(\frac{\alpha^{2F_{3k}}-\beta^{2F_{3k}}}{\sqrt{5}} + \frac{\alpha^{2F_{3k-1}}-\beta^{2F_{3k-1}}}{\sqrt{5}} + \frac{\alpha^{2F_{3k-2}}-\beta^{2F_{3k-2}}}{\sqrt{5}}\right)\\ &= \frac{1}{5}\left(F_{2F_{3k}} + F_{2F_{3k-1}} + F_{2F_{3k-2}}\right).\end{aligned}$$

Finally,

$$\sum_{k=1}^{n} F_{F_{3k}}F_{F_{3k-1}}F_{F_{3k-2}} = \frac{1}{5}\sum_{k=1}^{n}\left(F_{2F_{3k}} + F_{2F_{3k-1}} + F_{2F_{3k-2}}\right) = \frac{1}{5}\sum_{k=1}^{3n} F_{2F_k},$$

which proves (i).

The proof of (ii) proceeds in a similar manner. Using the Binet's formula for the Lucas numbers, we find

$$\begin{aligned} L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} &= \left(\alpha^{L_{3k}}+\beta^{L_{3k}}\right)\left(\alpha^{L_{3k-1}}+\beta^{L_{3k-1}}\right)\left(\alpha^{L_{3k-2}}+\beta^{L_{3k-2}}\right) \\ &= \alpha^{2L_{3k}}+(\alpha\beta)^{L_{3k-2}}\alpha^{2L_{3k-1}}+(\alpha\beta)^{L_{3k-1}}\alpha^{2L_{3k-2}}+(\alpha\beta)^{L_{3k}} \\ &\quad +(\alpha\beta)^{L_{3k}}+(\alpha\beta)^{L_{3k-1}}\beta^{2L_{3k-2}}+(\alpha\beta)^{L_{3k-2}}\beta^{2L_{3k-1}}+\beta^{2L_{3k}}. \end{aligned}$$

It is known that L_{3k-1} and L_{3k-2} are both odd, and L_{3k} is even for any integer k. Hence, $(\alpha\beta)^{L_{3k-1}}=(\alpha\beta)^{L_{3k-2}}=-1$, and $(\alpha\beta)^{L_{3k}}=1$. Thus,

$$\begin{aligned} L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} &= \left(\alpha^{2L_{3k}}+\beta^{2L_{3k}}\right)-\left(\alpha^{2L_{3k-1}}+\beta^{2L_{3k-1}}\right)-\left(\alpha^{2L_{3k-2}}+\beta^{2L_{3k-2}}\right)+2 \\ &= L_{2L_{3k}}-L_{2L_{3k-1}}-L_{2L_{3k-2}}+2 \\ &= (-1)^{L_{3k}}L_{2L_{3k}}+(-1)^{L_{3k-1}}L_{2L_{3k-1}}+(-1)^{L_{3k-2}}L_{2L_{3k-2}}+2, \end{aligned}$$

which proves that

$$\sum_{k=1}^{n} L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} = 2n+\sum_{k=1}^{3n}(-1)^{L_k}L_{2L_k}.$$

Editor's Note: Plaza quoted the general formulas for the products $F_{x_1}F_{x_2}F_{x_3}$ and $L_{x_1}L_{x_2}L_{x_3}$ in [2], and Davenport applied the following symmetric identities from [1]:

$$\begin{aligned} 5F_xF_yF_z &= F_{x+y+z}-(-1)^xF_{-x+y+z}-(-1)^yF_{x-y+z}-(-1)^zF_{x+y-z}, \\ L_xL_yL_z &= L_{x+y+z}+(-1)^xL_{-x+y+z}+(-1)^yL_{x-y+z}+(-1)^zL_{x+y-z}. \end{aligned}$$

References

[1] P. S. Bruckman, *Solution to Problem B-890*, The Fibonacci Quarterly, **38.5** (2000), 469–470.
[2] H. H. Ferns, *Products of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **7.1** (1969), 1–13.

Also solved by Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Ángel Plaza, Raphael Schumacher (student), and the proposer.

A Double Binomial Sum

B-1204 Proposed by Steve Edwards, Kennesaw State University, Marietta, GA. (Vol. 55.1, February 2017)

For non-negative integers n, express

$$A_n=\sum_{j=0}^{n}\frac{1}{2^{2j}}\sum_{i=0}^{n+j}\binom{n+j-i}{n-j}\binom{n+j}{i} \quad \text{and} \quad B_n=\sum_{j=0}^{n-1}\frac{1}{2^{2j+1}}\sum_{i=0}^{n+j}\binom{n+j-i}{n-j-1}\binom{n+j}{i}$$

in terms of Fibonacci numbers.

Solution by Hideyuki Ohtsuka, Saitama, Japan.

We use the well-known identity

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}=F_{n+1}.$$

We have

$$\begin{aligned} A_n &= \sum_{j=0}^{n} \frac{1}{2^{2j}} \sum_{i=0}^{n+j} \frac{(n+j-i)!}{(n-j)!\,(2j-i)!} \cdot \frac{(n+j)!}{i!\,(n+j-i)!} \\ &= \sum_{j=0}^{n} \frac{1}{2^{2j}} \sum_{i=0}^{n+j} \frac{(n+j)!}{(n-j)!\,(2j)!} \cdot \frac{(2j)!}{i!\,(2j-i)!} \\ &= \sum_{j=0}^{n} \frac{1}{2^{2j}} \binom{n+j}{n-j} \sum_{i=0}^{n+j} \binom{2j}{i} \;=\; \sum_{j=0}^{n} \frac{1}{2^{2j}} \binom{n+j}{n-j} \sum_{i=0}^{2j} \binom{2j}{i} \\ &= \sum_{j=0}^{n} \frac{1}{2^{2j}} \binom{n+j}{n-j} \cdot 2^{2j} \;=\; \sum_{k=0}^{n} \binom{2n-k}{k} \;=\; F_{2n+1}, \end{aligned}$$

and

$$\begin{aligned} B_n &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \sum_{i=0}^{n+j} \frac{(n+j-i)!}{(n-j-1)!\,(2j-i+1)!} \cdot \frac{(n+j)!}{i!\,(n+j-i)!} \\ &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \sum_{i=0}^{n+j} \frac{(n+j)!}{(n-j-1)!\,(2j+1)!} \cdot \frac{(2j+1)!}{i!\,(2j-i+1)!} \\ &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \binom{n+j}{n-j-1} \sum_{i=0}^{n+j} \binom{2j+1}{i} \\ &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \binom{n+j}{n-j-1} \sum_{i=0}^{2j+1} \binom{2j+1}{i} \\ &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \binom{n+j}{n-j-1} \cdot 2^{2j+1} \;=\; \sum_{k=0}^{n-1} \binom{2n-1-k}{k} \;=\; F_{2n}. \end{aligned}$$

Also solved by Brian Bradie, I. V. Fadek, Dmitry Fleischman, Jaroslav Seibert, and the proposer.

Power-Mean and Jensen's Inequalities

B-1205 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania; and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 55.1, February 2017)

Prove that

$$n^{m-1} \sum_{k=1}^{n} F_k^{2m} \geq F_n^m F_{n+1}^m$$

for any positive integers n and m.

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

It is enough to apply the power-mean arithmetic mean inequality to the sequence $\left(F_k^2\right)_{1\le k\le n}$, as follows:

$$\sqrt[m]{\frac{1}{n}\sum_{k=1}^{n} F_k^{2m}} \ge \frac{1}{n}\sum_{k=1}^{n} F_k^2 = \frac{F_nF_{n+1}}{n}.$$

It follows that

$$n^{m-1}\sum_{k=1}^{n} F_k^{2m} \ge F_n^m F_{n+1}^m.$$

Solution 2 by Henry Ricardo, New York Math Circle, Purchase, NY.

Noting that, for any positive integer m, the function $f(x) = x^m$ is convex on the interval $(0,\infty)$, and that $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$, we use Jensen's inequality to conclude that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(F_k^2\right) \ge f\left(\frac{1}{n}\sum_{k=1}^{n} F_k^2\right),$$

or

$$n^{m-1}\sum_{k=1}^{n} F_k^{2m} \ge (F_nF_{n+1})^m = F_n^m F_{n+1}^m.$$

Also solved by Maria Aristizabal (student), Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Wei-Kai Lai, Soumitra Mandal, Hideyuki Ohtsuka, Nicuşor Zlota, and the proposer.