ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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We would like to thank our readers for their submissions during during our editorship over the past sixteen years.

Please submit all new problem proposals and their solutions to DR. HARRIS KWONG Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY 14063, or by email at kwong@fredonia.edu.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed (preferably in LaTeX) on separate sheets. Solutions to problems in this issue must be received by May 15, 2017. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1196 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any integer n, prove that

$$\frac{L_{n+2}^5 - L_{n+1}^5 - L_{n-1}^5 - L_{n-2}^5}{L_{n+2}^3 - L_{n+1}^3 - L_{n-1}^3 - L_{n-2}^3} = 5 \cdot \frac{F_{n+2}^5 - F_{n+1}^5 - F_{n-1}^5 - F_{n-2}^5}{F_{n+2}^3 - F_{n+1}^3 - F_{n-1}^3 - F_{n-2}^3}.$$

B-1197 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Let a and b be positive real numbers. For any positive integer n, prove each of the following:

$$\text{(i)}\ \sum_{k=1}^{n} \frac{F_k^4}{aL_k + bF_k^2} > \frac{F_n^2 F_{n+1}^2}{a(L_{n+2} - 3) + bF_n F_{n+1}},$$

$$\text{(ii)}\ \sum_{k=1}^{n} \frac{F_k^4}{aF_{n+2} + bF_k - a} > \frac{F_n^2 F_{n+1}^2}{(an + b)(F_{n+2} - 1)}.$$

B-1198 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let c be a positive integer. The sequence $\{a_n\}$ is defined by, for $n \geq 1$,

$$a_{n+2} = a_n + 2c \quad \text{with} \quad a_1 = 1 \quad \text{and} \quad a_2 = 3.$$

Prove that

$$\text{(i)}\ \sum_{n=1}^{\infty} \tan^{-1} \frac{F_c}{F_{a_n+c}} = \frac{\pi}{4}, \text{ if } c \text{ is even;}$$

$$\text{(ii)}\ \sum_{n=1}^{\infty} \tan^{-1} \frac{L_c}{L_{a_n+c}} = \frac{\pi}{4}, \text{ if } c \text{ is odd.}$$

B-1199 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.

For any positive integer number k, the k-Fibonacci and k-Lucas sequences, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$ both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$ and $L_{k,0} = 2$; $F_{k,1} = k$. Prove that for all integers $m \geq 1$ and $r \geq 1$,

$$kr^{m+1} F_{k,m+1} = \sum_{i=0}^{m} r^i L_{k,i} + (kr - 2) \sum_{i=0}^{m+1} r^{i-1} F_{k,i}.$$

B-1200 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let H_n denote the nth harmonic number. Prove that

$$\text{(i)}\ \sum_{n=1}^{\infty} \frac{H_n F_n}{2^n} = \log\left(4\alpha^{12/\sqrt{5}}\right) \quad \text{and} \quad \text{(ii)}\ \sum_{n=1}^{\infty} \frac{H_n L_n}{2^n} = \log\left(64\alpha^{4\sqrt{5}}\right).$$

SOLUTIONS

Closure !

**B-1176 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 53.4, November 2015)**

Find a closed form expression for the sum

$$\sum_{k=0}^{n} L_{3^k}^3 + 2\sum_{k=0}^{n} L_{3^k}.$$

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Working from the Binet formula for L_n, $L_n = \alpha^n + \beta^n$, we find

$$L_n^3 = \alpha^{3n} + \beta^{3n} + 3(\alpha\beta)^n(\alpha^n + \beta^n) = L_{3n} + 3(-1)^n L_n.$$

Therefore,

$$L_{3^k}^3 = L_{3^{k+1}} + 3(-1)^{3^k} L_{3^k} = L_{3^{k+1}} - 3L_{3^k},$$

and

$$\sum_{k=0}^{n} L_{3^k}^3 + 2\sum_{k=0}^{n} L_{3^k} = \sum_{k=0}^{n} L_{3^{k+1}} - \sum_{k=0}^{n} L_{3^k} = L_{3^{n+1}} - L_1 = L_{3^{n+1}} - 1.$$

Also solved by Adnan Ali (student), Miguel Cidras, Charlie K. Cook, Kenneth B. Davenport, Steve Edwards, Dmitry Fleischman, John Hawkins and David Stone (jointly), Russell Jay Hendel, Harris Kwong, Junes Leondro, Henri RiCardo, Jaroslav Seibert, Itzal de Urioste, Dan Weiner, and the proposer.

Find Their Limits

**B-1177 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 53.4, November 2015)**

For any integer $p \geq 0$, prove each of the following:

(i) $\displaystyle\lim_{n\to\infty} \frac{F_n^p + F_{n+3}^p}{F_{n+2}^p} = \frac{L_{2p} + L_p}{2} - \frac{F_{2p} - F_p}{2}\sqrt{5}$

(ii) $\displaystyle\lim_{n\to\infty} \frac{L_n^p + L_{n+3}^p}{L_{n+2}^p} = \frac{L_{2p} + L_p}{2} - \frac{F_{2p} - F_p}{2}\sqrt{5}.$

Solution by Jaroslav Seibert, Institute of Mathematics and Quantitative Methods, Faculty of Economics and Administration, University of Pardubice, Czech Republic.

We will consider the generalized Fibonacci numbers defined by the recurrence relation $G_{n+2} = G_{n+1} + G_n$ with arbitrary initial terms G_0, G_1.

It is known that $\lim_{n\to\infty} \dfrac{G_{n+k}}{G_n} = \alpha^k$ (e.g.[1, p. 247]) for each integer k. Then the following holds for the limit on the left-hand side of the given equalities

$$\lim_{n\to\infty} \frac{G_n^p + G_{n+3}^p}{G_{n+2}^p} = \lim_{n\to\infty} \left(\frac{G_n}{G_{n+2}}\right)^p + \lim_{n\to\infty} \left(\frac{G_{n+3}}{G_{n+2}}\right)^p = \left(\frac{1}{\alpha^2}\right)^p + \alpha^p = \frac{1}{\alpha^{2p}} + \alpha^p = \frac{1+\alpha^{3p}}{\alpha^{2p}}.$$

The expressions on the right-hand side of (i) and (ii) can be rewritten using the Binet formula and the relation $\alpha\beta = -1$, as

$$\begin{aligned}
\frac{L_{2p}+L_p}{2} - \frac{F_{2p}-F_p}{2}\sqrt{5} &= \frac{\alpha^{2p}+\beta^{2p}+\alpha^p+\beta^p}{2} - \frac{\alpha^{2p}-\beta^{2p}-\alpha^p+\beta^p}{2\sqrt{5}}\sqrt{5} \\
&= \beta^{2p} + \alpha^p \\
&= \left(-\frac{1}{\alpha}\right)^{2p} + \alpha^p \\
&= \frac{1}{\alpha^{2p}} + \alpha^p \\
&= \frac{1+\alpha^{3p}}{\alpha^{2p}}.
\end{aligned}$$

It follows that the equality $\lim_{n\to\infty} \dfrac{G_n^p + G_{n+3}^p}{G_{n+2}^p} = \dfrac{L_{2p}+L_p}{2} - \dfrac{F_{2p}-F_p}{2}\sqrt{5}$ holds for the generalized Fibonacci numbers G_n, and in particular, for the Fibonacci and Lucas numbers.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by Adnan Ali (student), Brian D. Beasley, Brian Bradie, Miguel Cidras, Charlie K. Cook and Michael R. Bacon (jointly), Kenneth B. Davenport, Steve Edwards, Dmitry Fleischman, John Hawkins and David Stone (jointly), Hideyuki Ohtsukas, Dan Weiner, and the proposer.

The Ubiquitous AM-GM Inequality

B-1178 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 53.4, November 2015)

Prove that

$$\sqrt{F_1^4 - F_1^2F_2^2 + F_2^4} + \sqrt{F_2^4 - F_2^2F_3^2 + F_3^4} + \cdots \\ + \sqrt{F_{n-1}^4 - F_{n-1}^2F_n^2 + F_n^4} + \sqrt{F_n^4 - F_n^2 + 1} > F_nF_{n+1}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

Given any real numbers x and y, the AM-GM inequality asserts that $x^4+y^4 \geq 2x^2y^2$. Then

$$\begin{aligned}4(x^4-x^2y^2+y^4) &= x^4+y^4+3(x^4+y^4)-4x^2y^2\\ &\geq x^4+y^4+6x^2y^2-4x^2y^2\\ &= (x^2+y^2)^2.\end{aligned}$$

Hence,

$$\sqrt{x^4-x^2y^2+y^4} \geq \frac{x^2+y^2}{2},$$

with equality holds only when $x^2=y^2$. Take note that $F_n^4-F_n^2+1=F_n^4-F_n^2F_1^2+F_1^4$. The sum S on the left-hand side of the given inequality is a cyclic sum. Therefore,

$$S > \sum_{i=1}^{n} F_i^2 = F_nF_{n+1}.$$

Remark: It is clear that the argument can also be applied to the Lucas numbers. Since $L_1=1$, and $\sum_{i=1}^{n} L_i^2 = L_nL_{n+1}-2$, we find the analogous result

$$\begin{aligned}&\sqrt{L_1^4-L_1^2L_2^2+L_2^4}+\sqrt{L_2^4-L_2^2L_3^2+L_3^4}+\cdots\\ &\quad+\sqrt{L_{n-1}^4-L_{n-1}^2L_n^2+L_n^4}+\sqrt{L_n^4-L_n^2+1} > L_nL_{n+1}-2.\end{aligned}$$

Also solved by Adnan Ali (student), Kenneth B. Davenport, Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Jaroslav Seibert, Itzal Urioste, Nicuşor Zlota, and the proposer.

Sine, Cosine, and a Fibonacci Inequality

B-1179 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**
(Vol. 53.4, November 2015)

Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that

$$2 \cdot \left(\sum_{k=1}^{n} F_k \cdot \sin x_k\right) \cdot \left(\sum_{k=1}^{n} F_k \cdot \cos x_k\right) \leq n \cdot F_n \cdot F_{n+1}.$$

Solution by Adnan Ali, (student), A.E.C.S-4, Mumbai, India.

From the AM-GM and Cauchy-Schwartz Inequalities, we have

$$\begin{aligned}
2\cdot\left(\sum_{k=1}^{n} F_k\cdot\sin x_k\right)\cdot\left(\sum_{k=1}^{n} F_k\cdot\cos x_k\right) &\le \left(\sum_{k=1}^{n} F_k\cdot\sin x_k\right)^2+\left(\sum_{k=1}^{n} F_k\cdot\cos x_k\right)^2\\
&\le \left(\sum_{k=1}^{n} F_k^2\right)\left(\sum_{k=1}^{n}\sin^2 x_k\right)+\left(\sum_{k=1}^{n} F_k^2\right)\left(\sum_{k=1}^{n}\cos^2 x_k\right)\\
&= n\left(\sum_{k=1}^{n} F_k^2\right) n\cdot F_n\cdot F_{n+1}.
\end{aligned}$$

Also solved by Brian Bradie, Miguel Cidras, Kenneth B. Davenport, Steve Edwards, Dmitry Fleischman, Harris Kwong, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Henri RiCardo, Jaroslav Seibert, Itzal de Urioste, Dan Weiner, Nicuşor Zlota, and the proposer.

A Complex One!

B-1180 Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 53.4, November 2015)

If $i=\sqrt{-1}$, find

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n}+i}.$$

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

We use the identities $F_{2n}^2+1=F_{2n-1}F_{2n+1}$ and $F_{2n+2}F_{2n-1}-F_{2n}F_{2n+1}=1$, which can be found in [1].

For the summand we have

$$\begin{aligned}
\frac{1}{F_{2n}+i} &= \frac{1}{F_{2n}+i}\cdot\frac{F_{2n}-i}{F_{2n}-i}\\
&= \frac{F_{2n}}{F_{2n}^2+1}-i\frac{1}{F_{2n}^2+1}\\
&= \frac{F_{2n+1}-F_{2n-1}}{F_{2n-1}F_{2n+1}}-i\frac{F_{2n+2}F_{2n-1}-F_{2n}F_{2n+1}}{F_{2n-1}F_{2n+1}}\\
&= \left(\frac{1}{F_{2n-1}}-\frac{1}{F_{2n-1}}\right)-i\left(\frac{F_{2n+2}}{F_{2n+1}}-\frac{F_{2n}}{F_{2n-1}}\right).
\end{aligned}$$

The sum is telescoping, and the kth partial sum is

$$\left(\frac{1}{F_{-1}}-\frac{1}{F_{2k+1}}\right)-i\left(\frac{F_{2k+2}}{F_{2k+1}}-\frac{F_0}{F_1}\right),$$

and since $\lim\limits_{n\to\infty}\dfrac{F_{n+1}}{F_n}=\alpha$, the series converges to $1-\alpha i$.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Halsted Press, 1989.

Also solved by Adnan Ali (student), Brian Beasley, Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Harris Kwong, Poo-Sung Park, Ángel Plaza, Jaroslav Seibert, Dan Weiner, and the proposer.

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