ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2019. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1236 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that, for any integers $m \ge 0$ and $n > 1$,

$$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}^{m+1}}{F_k^{2m}} > \frac{2^{n(m+1)}}{F_{n+1}^m F_{n+2}^m}, \qquad \text{and} \qquad \sum_{k=1}^{n+1} \frac{F_k^{m+1}}{\binom{n}{k-1}^m} > \frac{(F_{n+3}-1)^{m+1}}{2^{mn}}.$$

B-1237 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Evaluate

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{\alpha^k + \alpha}\right), \qquad \text{and} \qquad \prod_{k=1}^{\infty}\left(1 - \frac{1}{\alpha^k + \alpha}\right).$$

B-1238 **Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

Let $a > 1$ and consider the sequence of real numbers defined recursively by $x_0 = 0$, $x_1 = 1$, and

$$x_{n+1} = \left(a + \frac{1}{a}\right) x_n - x_{n-1}, \qquad n \geq 1.$$

Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{x_{2^n}}$ is a rational number if and only if a is a rational number.

B-1239 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For all integers n, prove that

$$\left(\frac{1}{L_n} - \frac{1}{L_{n+1}}\right)^4 + \left(\frac{1}{L_{n+1}} + \frac{1}{L_{n+2}}\right)^4 + \left(\frac{1}{L_{n+2}} + \frac{1}{L_n}\right)^4 = 2\left(\frac{1}{L_n} + \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}}\right)^4.$$

B-1240 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

Assume $x_k > 0$ for $k = 1, 2, \ldots, n$. Prove that, for any positive integers $m \geq 1$ and $n > 1$,

$$\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{\substack{i=1 \\ \text{cyclic}}}^{n} \frac{x_i x_{i+1}}{F_m x_i + F_{m+1} x_{i+1}}\right) \geq \frac{n^2}{F_{m+2}},$$

$$\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{\substack{i=1 \\ \text{cyclic}}}^{n} \frac{x_i x_{i+1}}{L_m x_i + L_{m+1} x_{i+1}}\right) \geq \frac{n^2}{L_{m+2}}.$$

SOLUTIONS

Editor's Notes. In the solution to Elementary Problem B-1208 that appeared in the May issue, the first round of row reductions should be carried out according to $k = n + 1, n, \ldots, 3$. The two rounds of row reductions can be combined into one. For $k = n + 1, n, \ldots, 3$, subtract the sum of row $k - 1$ and row $k - 2$ from row k. Next, subtracting the first row from the second yields the last augmented matrix shown in the solution.

Another Application of the AM-GM Inequality

B-1216 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**
(Vol. 55.4, November 2017)

Prove that, for any positive real number m, and any positive integer n,

$$F_n^m F_{n+1}^m \sum_{k=1}^{n} \frac{L_k^{m+1}}{F_k^{2m}} \geq n^{m+1} \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}}.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The proposed inequality follows from the AM-GM inequality and the identity $F_n F_{n+1} = \sum_{k=1}^{n} F_k^2$:

$$\begin{aligned} F_n^m F_{n+1}^m \sum_{k=1}^{m} \frac{L_k^{m+1}}{F_k^{2m}} &\geq F_n^m F_{n+1}^n \cdot n \sqrt[n]{\prod_{k=1}^{n} \frac{L_k^{m+1}}{F_k^{2m}}} \\ &= \left(\frac{F_n F_{n+1}}{\sqrt[n]{\prod_{k=1}^{n} F_k^2}} \right)^m \cdot n \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}} \\ &= \left(\frac{\sum_{k=1}^{n} F_k^2}{\sqrt[n]{\prod_{k=1}^{n} F_k^2}} \right)^m \cdot n \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}} \\ &\geq \left(\frac{n \sqrt[n]{\prod_{k=1}^{n} F_k^2}}{\sqrt[n]{\prod_{k=1}^{n} F_k^2}} \right)^m \cdot n \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}} \\ &= n^{m+1} \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}}. \end{aligned}$$

Solution 2 by Wei-Kai Lai and John Risher (student) (jointly), University of South Carolina Salkehatchie, Walterboro, SC.

According to Radon's Inequality, we know that

$$\sum_{k=1}^{n} \frac{L_k^{m+1}}{F_k^{2m}} \geq \frac{\left(\sum_{k=1}^{n} L_k\right)^{m+1}}{\left(\sum_{k=1}^{n} F_k^2\right)^m}.$$

To prove the claimed inequality, we therefore only need to prove that

$$F_n^m F_{n+1}^m \frac{\left(\sum_{k=1}^{n} L_k\right)^{m+1}}{\left(\sum_{k=1}^{n} F_k^2\right)^m} \geq n^{m+1} \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}}.$$

Since $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, the above inequality is equivalent to

$$\left(\sum_{k=1}^{n} L_k \right)^{m+1} \geq n^{m+1} \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}},$$

which is apparently true due to the AM-GM inequality.

Also solved by Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Hideyuki Ohtsuka, and the proposers.

Help From Exponential Generating Function

B-1217 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 55.4, November 2017)

Let $M_{k_i} = 2^{(i-1)k_i} L_{k_i}$. For integers $r \ge 1$ and $n \ge 0$, find a closed form expression for the sum

$$S_n = \sum_{\substack{0 \le k, k_1, \ldots, k_r \le n \\ k + k_1 + \cdots + k_r = n}} \frac{F_k M_{k_1} M_{k_2} \cdots M_{k_r}}{k!\, k_1!\, k_2!\, \cdots\, k_r!}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

The exponential generating function for the Fibonacci numbers is

$$G_F(x) = \sum_{k=0}^{\infty} \frac{F_k}{k!}\, x^k = \frac{1}{\sqrt{5}} \left(e^{\alpha x} - e^{\beta x} \right),$$

whereas the exponential generating function for the Lucas numbers is

$$G_L(x) = \sum_{k=0}^{\infty} \frac{L_k}{k!}\, x^k = e^{\alpha x} + e^{\beta x}.$$

It follows that the exponential generating function for M_{k_i} is

$$G_i(x) = \sum_{k_i=0}^{\infty} \frac{M_{k_i}}{k_i!}\, x^{k_i} = \sum_{k_i=0}^{\infty} \frac{L_{k_i}}{k_i!} \left(2^{i-1} x \right)^{k_i} = G_L(2^{i-1}x) = e^{2^{i-1}\alpha x} + e^{2^{i-1}\beta x}.$$

Due to convolution, we can now recognize S_n as the coefficient of x^n in the product

$$\begin{aligned} & G_F(x) G_1(x) G_2(x) \cdots G_r(x) \\ = \;& \tfrac{1}{\sqrt{5}} \left(e^{\alpha x} - e^{\beta x} \right) \left(e^{\alpha x} + e^{\beta x} \right) \left(e^{2\alpha x} + e^{2\beta x} \right) \cdots \left(e^{2^{r-1}\alpha x} + e^{2^{r-1}\beta x} \right) \\ = \;& \tfrac{1}{\sqrt{5}} \left(e^{2\alpha x} - e^{2\beta x} \right) \left(e^{2\alpha x} + e^{2\beta x} \right) \cdots \left(e^{2^{r-1}\alpha x} + e^{2^{r-1}\beta x} \right) \\ = \;& \tfrac{1}{\sqrt{5}} \left(e^{4\alpha x} - e^{4\beta x} \right) \cdots \left(e^{2^{r-1}\alpha x} + e^{2^{r-1}\beta x} \right) \\ \vdots \;& \quad \vdots \\ = \;& \tfrac{1}{\sqrt{5}} \left(e^{2^r \alpha x} - e^{2^r \beta x} \right). \end{aligned}$$

Therefore,

$$S_n = \frac{1}{\sqrt{5}} \left[\frac{(2^r \alpha)^n}{n!} - \frac{(2^r \beta)^n}{n!} \right] = \frac{2^{rn} F_n}{n!}.$$

Also solved by Raphael Schumacher (student), and the proposer.

Simplifying a Complicated Expression

B-1218 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 55.4, November 2017)

Find a closed form expression for

$$(L_{n+1}-1)F_n(F_{2n+2}-F_{n+2})+(1-F_n-F_{n+2})F_{n+2}(F_{2n+2}-F_{n+3})+(F_{2n+2}-F_{n+2})(F_{2n+2}-F_{n+3}).$$

Solution 1 by Charles K. Cook, Sumter, SC.

The well-known identities $F_{2n} = F_nL_n$ and $L_n = F_{n-1} + F_{n+1}$ will be used as needed. Let A represent the first term, B, the second, and C, the third, of the given sum. Expanding and using the above identities yields

$$\begin{aligned} A &= F_nF_{n+1}L_{n+1}^2 - F_n(F_{n+1} + F_{n+2})L_{n+1} + F_nF_{n+2}, \\ B &= -F_{n+1}F_{n+2}L_{n+1}^2 + F_{n+2}(F_{n+1} + F_{n+3})L_{n+1} - F_{n+2}F_{n+3}, \\ C &= F_{n+1}^2L_{n+1}^2 - F_{n+1}(F_{n+2} + F_{n+3})L_{n+1} + F_{n+2}F_{n+3}. \end{aligned}$$

The coefficient for L_{n+1}^2 in the sum is

$$F_nF_{n+1} - F_{n+1}F_{n+2} + F_{n+1}^2 = F_{n+1}(F_n - F_{n+2} + F_{n+1}) = 0,$$

whereas the coefficient for L_{n+1} is

$$\begin{aligned} &-F_n(F_{n+1} + F_{n+2}) + F_{n+2}(F_{n+1} + F_{n+3}) - F_{n+1}(F_{n+2} + F_{n+3}) \\ &\quad = -F_n(F_{n+1} + F_{n+2}) + (F_{n+2} - F_{n+1})F_{n+3} \\ &\quad = 0, \end{aligned}$$

and the remaining terms are

$$F_nF_{n+2} - F_{n+2}F_{n+3} + F_{n+2}F_{n+3} = F_nF_{n+2}.$$

Thus, adding A, B, and C, the required closed form for the given sum is F_nF_{n+2}.

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

We use the well-known identities $F_{2m} = F_mL_m$, and $F_{m-1} + F_{m+1} = L_m$. Let $t = L_{n+1} - 1$. Then, we have

$$\begin{gathered} F_{2n+2} - F_{n+2} = F_{n+1}L_{n+1} - F_{n+1} - F_n = tF_{n+1} - F_n; \\ F_{2n+2} - F_{n+3} = F_{n+1}L_{n+1} - F_{n+1} - F_{n+2} = tF_{n+1} - F_{n+2}; \\ 1 - F_n - F_{n+2} = 1 - L_{n+1} = -t. \end{gathered}$$

By the above identities, the expression of the problem is

$$\begin{aligned} &tF_n(tF_{n+1} - F_n) - tF_{n+2}(tF_{n+1} - F_{n+2}) + (tF_{n+1} - F_n)(tF_{n+1} - F_{n+2}) \\ &\quad = t^2F_{n+1}(F_n - F_{n+2} + F_{n+1}) + t\left[F_{n+2}(F_{n+2} - F_{n+1}) - F_n(F_n + F_{n+1})\right] + F_nF_{n+2} \\ &\quad = t(F_{n+2}F_n - F_nF_{n+2}) + F_nF_{n+2} \\ &\quad = F_nF_{n+2}. \end{aligned}$$

Solution 3 by the proposer.

We use the identity $F_{2n+2} = F_{n+1}L_{n+1} = F_{n+1}(F_n + F_{n+2})$ to write the given expression as

$$\begin{aligned} F_nF_{n+2}\Bigg[&\frac{(F_{2n+2} - F_{n+1})(F_{2n+2} - F_{n+2})}{F_{n+1}F_{n+2}} \\ &- \frac{(F_{2n+2} - F_{n+1})(F_{2n+2} - F_{n+3})}{F_nF_{n+1}} + \frac{(F_{2n+2} - F_{n+2})(F_{2n+2} - F_{n+3})}{F_nF_{n+2}}\Bigg]. \end{aligned}$$

Let

$$P(x) = \frac{(x - F_{n+1})(x - F_{n+2})}{F_{n+1}F_{n+2}} - \frac{(x - F_{n+1})(x - F_{n+3})}{F_nF_{n+1}} + \frac{(x - F_{n+2})(x - F_{n+3})}{F_nF_{n+2}}.$$

We have

$$P(F_{n+1}) = P(F_{n+2}) = P(F_{n+3}) = 1.$$

Therefore, $P(x) \equiv 1$. Thus, a closed form for the expression is

$$F_n F_{n+2} \cdot P(F_{2n+2}) = F_n F_{n+2}.$$

Also solved by Brian D. Beasley, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, G. C. Greubel, Kantaphon Kuhapatanakul, Wei-Kai Lai, Ehren Metcalfe, Verónica Molina Reales (student), Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, Elizabeth S. Spoehel (student), and the proposers.

An Inequality with a Cyclic Sum

B-1219 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 55.4, November 2017)

Prove that, for any integer $n \geq 2$,

$$\frac{F_n^4 + F_n^2 + 1}{F_n} + \sum_{k=1}^{n-1} \frac{F_k^4 + F_k^2 F_{k+1}^2 + F_{k+1}^4}{F_k F_{k+1}} > 3F_n F_{n+1}.$$

Editor's Note: The condition on n should be $n \geq 3$.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since $F_1 = 1$, and

$$F_n F_{n+1} = \sum_{k=1}^{n} F_k^2,$$

the proposed inequality may be written as

$$\sum_{\substack{k=1 \\ \text{cyclic}}}^{n} \frac{F_k^4 + F_k^2 F_{k+1}^2 + F_{k+1}^4}{F_k F_{k+1}} > 3 \sum_{k=1}^{n} F_k^2,$$

which is a special case of the following more general inequality.

Lemma. *Let $a_1, \ldots, a_m$ be a sequence of positive real numbers. Then,*

$$\sum_{\substack{k=1 \\ cyclic}}^{m} \frac{a_k^4 + a_k^2 a_{k+1}^2 + a_{k+1}^4}{a_k a_{k+1}} \geq 3 \sum_{k=1}^{m} a_k^2.$$

Proof. It is enough to prove that, if $a, b > 0$, then

$$\frac{a^4 + a^2 b^2 + b^4}{ab} \geq \frac{3}{2}\,(a^2 + b^2),$$

which is equialent to

$$2(a^4 + a^2 b^2 + b^4) \geq 3ab(a^2 + b^2).$$

To complete the proof, observe that

$$a^4 + b^4 \geq a^3 b + ab^3 = ab(a^2 + b^2),$$
$$a^4 + 2a^2 b^2 + b^4 = (a^2 + b^2)(a^2 + b^2) \geq 2ab(a^2 + b^2).$$

To obtain a strict inequality, we need $m \geq 2$, and some of the terms in the sequence $a_1, \ldots, a_m$ have to be different. □

Notice that the inequality in the problem becomes an identity when $n = 2$.

Also solved by Brian D. Beasley, Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Wei-Kai Lai and John Risher (student) (jointly), Hideyuki Ohtsuka, and the proposers.

Gelin-Cesàro Identity Yields a Telescoping Product

B-1220 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 55.4, November 2017)

Prove that

$$\prod_{n=3}^{\infty} \left(1 - \frac{1}{F_n^4}\right) = \frac{\alpha^5}{12}.$$

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

Using the Gelin-Cesàro Identity $F_n^4 - 1 = F_{n-2}F_{n-1}F_{n+1}F_{n+2}$, we have

$$1 - \frac{1}{F_n^4} = \frac{F_n^4 - 1}{F_n^4} = \frac{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}{F_n^4}.$$

It follows from the telescoping property that, for $m \geq 4$,

$$\prod_{n=3}^{m} \left(1 - \frac{1}{F_n^4}\right) = \prod_{n=3}^{m} \frac{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}{F_n^4} = \frac{F_1 F_2^2}{F_3^2 F_4} \cdot \frac{F_{m+1}^2 F_{m+2}}{F_{m-1}F_m^2} = \frac{F_{m+1}^2 F_{m+2}}{12 F_{m-1} F_m^2}.$$

Since $\lim_{m\to\infty} F_{m+j}/F_m = \alpha^j$, we find

$$\prod_{n=3}^{\infty} \left(1 - \frac{1}{F_n^4}\right) = \lim_{m\to\infty} \frac{F_{m+1}^2 F_{m+2}}{12 F_{m-1} F_m^2} = \lim_{m\to\infty} \frac{1}{12} \left(\frac{F_{m+1}}{F_m}\right)^2 \frac{F_{m+2}}{F_{m-1}} = \frac{\alpha^2 \cdot \alpha^3}{12} = \frac{\alpha^5}{12}.$$

Also solved by Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Kantaphon Kuhapatanakul, Ángel Plaza, Raphael Schumacher (student), and the proposer.