ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2010. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1047 (Correction) **Proposed by Charles K. Cook, Sumter, SC**

Given positive integers n, p, and q greater than 1, show that

$$\sqrt[pq]{F_n^{p+q}} + \sqrt[pq]{L_n^{p+q}} > \sqrt[p]{F_n}\sqrt[q]{L_n} + \sqrt[p]{L_n}\sqrt[q]{F_n}.$$

B-1051 **Proposed by Charles K. Cook, Sumter, SC**

For all positive integers n show that $F_n^4 + L_n^4 - 6F_{2n} + 5 > 0$.

B-1052 **Proposed by Br. J. Mahon, Australia**

Prove that

$$\sum_{r=2}^{\infty} \frac{F_r^2 + (-1)^r r^2}{F_{r+1}^{(1)} F_r^{(1)}} = \frac{5}{\alpha}$$

where $\{F_n^{(1)}\}$ is the sequence of first order convolutions of the Fibonacci numbers defined by

$$F_n^{(1)} = \sum_{i=0}^{n} F_{n-i} F_i.$$

B-1053 **Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politècnica de Cataluña, Barcelona, Spain**

Let n be a nonnegative integer. Prove that

$$\frac{1}{F_{n+2}} \left(\sqrt[3]{F_n F_{n+1}} + \sqrt[3]{F_{n+2} F_{n+3}} \right) < \sqrt{6}.$$

B-1054 **Proposed by H.- J. Seiffert, Berlin, Germany**

Show that the sequence $\{x_n\}_{n \geq 1}$ defined recursively by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{F_n x_n + F_{n+1}}{F_n x_n + F_{n-1}} \quad \text{for} \quad n \geq 1,$$

converges and find the limit.

B-1055 **Proposed by G. C. Greubel, Newport News, VA**

Find all integer solutions to the equation

$$x^2 + 6xy + 4y^2 = 4.$$

SOLUTIONS

Lucas and Fibonacci Squares

B-1044 **Proposed by Paul S. Bruckman, Sointula, Canada**
(Vol. 46/47.1, February 2008/2009)

Prove the following identities:

(1) $(L_n)^2 = 2(F_{n+1})^2 - (F_n)^2 + 2(F_{n-1})^2$;

(2) $25(F_n)^2 = 2(L_{n+1})^2 - (L_n)^2 + 2(L_{n-1})^2$.

Solution by Jaroslav Seibert, University Pardubice, The Czech Republic

We will consider the generalized Fibonacci numbers G_n given by the basic recurrence $G_{n+2} = G_{n+1} + G_n$ with arbitrary initial terms G_0, G_1. Then we can write

$$\begin{aligned} 2G_{n+1}^2 - G_n^2 + 2G_{n-1}^2 &= (G_{n+1} + G_{n-1})^2 + G_{n+1}^2 - G_n^2 + G_{n-1}^2 - 2G_{n+1}G_{n-1} \\ &= (G_{n+1} + G_{n-1})^2 + (G_{n+1} - G_{n-1})^2 - G_n^2 \\ &= (G_{n+1} + G_{n-1})^2 + G_n^2 - G_n^2 = (G_{n+1} + G_{n-1})^2. \end{aligned}$$

Using the formula $F_{n-1} + F_{n+1} = L_n$ ([1], identity (6)) we obtain the first equality and using the formula $L_{n-1} + L_{n+1} = 5F_n$ ([1], identity (5)) we obtain the second equality.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Chichester, Ellis Horwood Ltd., 1989.

Also solved by Brian Beasley, Scott H. Brown, Charles K. Cook, Kenneth B. Davenport, Steve Edwards, G. C. Greubel, Ralph Grimaldi, Russell J. Hendel, Rebecca A. Hillman, Harris Kwong, Graham Lord, John F. Morrison, Maitland Rose, H.- J. Seiffert, James Sellers, Omprakash Sikhwal, and the proposer.

Sum of a Product

B-1045 Proposed by H.- J. Seiffert, Berlin, Germany (Vol. 46/47.1, February 2008/2009)

Show that, for all positive integers n,

$$\sum_{k=1}^{n} 4^{n-k} \frac{F_k}{F_{k+1}} \left(\prod_{j=k}^{n} \frac{F_j}{L_j} \right)^2 = \frac{F_n}{F_{n+1}}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Let S_n denote the summation on the left. It is easy to verify that

$$S_n = \left(4S_{n-1} + \frac{F_n}{F_{n+1}} \right) \left(\frac{F_n}{L_n} \right)^2$$

for all $n \geq 2$. It is clear that $S_1 = 1 = F_1/F_2$. Assume $S_n = F_n/F_{n+1}$ for some integer $n \geq 1$. Then

$$S_{n+1} = \left(4S_n + \frac{F_{n+1}}{F_{n+2}} \right) \left(\frac{F_{n+1}}{L_{n+1}} \right)^2 = \frac{4F_nF_{n+2} + F_{n+1}^2}{F_{n+1}F_{n+2}} \left(\frac{F_{n+1}}{L_{n+1}} \right)^2.$$

Simpson's formula asserts that $F_nF_{n+2} - F_{n+1}^2 = (-1)^{n+1}$. Together with $L_{n+1}^2 - 5F_{n+1}^2 = 4(-1)^{n+1}$, we obtain $4F_nF_{n+2} = F_{n+1}^2 = L_{n+1}^2$. Therefore, $S_{n+1} = F_{n+1}/F_{n+2}$. It follows from induction that $S_n = F_n/F_{n+1}$ for all integers $n \geq 1$.

Also solved by Paul S. Bruckman, Steve Edwards, G. C. Greubel, Jaroslav Seibert, and the proposer.

A Lot of Zeros!

B-1046 Proposed by Michael Jemison (student), Northwest Missouri State University, Maryville, MO (Vol. 46/47.1, February 2008/2009)

Prove or disprove the following statement:
For $n \geq 1$,

$$F_{75(10^n)} \equiv 0 \pmod{10^{n+2}}.$$

Solution by Russell J. Hendel, Towson University, Towson, MD

The problem statement follows from the following 3 results found in [1]:

$$2^{n+2} | F_{3 \cdot 2^n}, \quad n \geq 1, \tag{1}$$

$$F_m | F_{am}, \quad m, a \geq 1. \tag{2}$$

$$\text{If} \quad p | F_q, \quad \text{then} \quad p^k | F_{qp^{k-1}}, \quad p, q, k \geq 1. \tag{3}$$

Letting $a = 5^{n+2}, m = 3 \cdot 2^n$ in (2) and using (1) we have

$$2^{n+2} | F_{3 \cdot 2^n} | F_{3 \cdot 5^{n+2} \cdot 2^n} = F_{75 \cdot 10^n}. \tag{4}$$

Letting $p = 5 = q$ and $k = n + 2$ in (3), and letting $a = 3 \cdot 2^n$, $m = 5^{n+2}$ in (2) we have

$$5^{n+2} | F_{5^{n+2}} | F_{3 \cdot 2^n \cdot 5^{n+2}} = F_{75 \cdot 10^n}, \quad n \geq 1. \tag{5}$$

Combining (4) and (5) yields the problem result.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, NY, 2001.

Also solved by Paul S. Bruckman, G. C. Greubel, and Jaroslav Siebert.