ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at REuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2009. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1047 Proposed by Charles K. Cook, Sumter, SC

Given positive integers n, p, and q greater than 1, show that

$$\sqrt[pq]{F_n^{p+q}} + \sqrt[pq]{F_n^{p+q}} > \sqrt[p]{F_n}\sqrt[q]{L_n} + \sqrt[p]{L_n}\sqrt[q]{F_n}.$$

B-1048 Proposed by José Luis Díaz-Barrero, Universidad de Cataluña, Barcelona, Spain

Let n be a positive integer. Prove that

$$\tan\left(\frac{2F_{n-1}}{1+L_n^2}\right) \le \frac{2F_{n-1}}{1+F_{2n}} \le \tan\left(\frac{2F_{n-1}}{1+F_n^2}\right).$$

B-1049 Proposed by Paul S. Bruckman, Sointula, Canada

Prove the following identities:

(1) $2(L_n)^3 = -(F_{n+2})^3 + 9(F_{n+1})^3 + 9(F_{n-1})^3 + (F_{n-2})^3$;

(2) $250(F_n)^3 = -(L_{n+2})^3 + 9(L_{n+1})^3 + 9(L_{n-1})^3 + (L_{n-2})^3$.

B-1050 Proposed by R. David Mitchell, University of South Carolina Sumter, Sumter, SC

For the generalized Fibonacci sequence $ja_{n-1} + ka_n = a_{n+1}, n \geq 2$, where j, k, a_1, and a_2 are non-zero integers, find functions $f(j,k,a_1,a_2)$ and $g(j,n)$ such that $a_n^2 - a_{n-1}a_{n+1} = f(j,k,a_1,a_2)g(j,n)$.

SOLUTIONS

Three Vanishing Sums

B-1040 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 45.4, November 2007)

If $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the standard *Fibonomial coefficient*, prove the following identities, valid for $m = 0, 1, 2, \ldots$.

(a) $\displaystyle\sum_{k=0}^{2m} (-1)^k \begin{bmatrix} 4m+1 \\ 2k \end{bmatrix} F_{2k} = 0.$

(b) $\displaystyle\sum_{k=0}^{2m+1} (-1)^k \begin{bmatrix} 4m+3 \\ 2k+1 \end{bmatrix} F_{2k+1} = 0.$

(c) $\displaystyle\sum_{k=0}^{4m} (-1)^{k(k+1)/2} \begin{bmatrix} 4m \\ k \end{bmatrix} F_k = 0.$

Solution by Russell J. Hendel, Towson University, Towson, MD

(a) Let $x = 4m+1$. We prove the stronger assertion

$$\begin{bmatrix} x \\ 2k \end{bmatrix} = \begin{bmatrix} x \\ x-(2k-1) \end{bmatrix}, \tag{1}$$

from which it immediately follows that an alternating sum would vanish. To prove (1) note that

$$\begin{aligned}
\begin{bmatrix} x \\ 2k \end{bmatrix} F_{2k} &= \frac{F_x F_{x-1} \cdots F_{x-2k+1}}{F_1 F_2 \cdots F_{2k-1} F_{2k}} F_{2k}, \text{ by definition of the Fibonomial coefficient,} \\
&= \frac{F_x F_{x-1} \cdots F_{x-2k+2}}{F_1 F_2 \cdots F_{2k-1}} F_{x-2k+1}, \text{ by cancellation and rearrangement,} \\
&= \begin{bmatrix} x \\ 2k-1 \end{bmatrix} F_{x-2k+1}, \text{ by definition of the Fibonomial coefficient,} \\
&= \begin{bmatrix} x \\ x-(2k-1) \end{bmatrix} F_{x-2k+1}, \text{ by the symmetry property of Fibonomial coefficients.}
\end{aligned}$$

(b) Similarly, letting $x = 4m+3$ we prove the stronger assertion that

$$\begin{aligned}
\begin{bmatrix} x \\ 2k+1 \end{bmatrix} F_{2k+1} &= \frac{F_x F_{x-1} \cdots F_{x-2k}}{F_1 F_2 \cdots F_{2k-1} F_{2k+1}} F_{2k+1}, \text{ by definition of the Fibonomial coefficient,} \\
&= \frac{F_x F_{x-1} \cdots F_{x-2k-1}}{F_1 F_2 \cdots F_{2k}} F_{x-2k}, \text{ by cancellation and rearrangement,} \\
&= \begin{bmatrix} x \\ 2k \end{bmatrix} F_{x-2k}, \text{ by definition of the Fibonomial coefficient,} \\
&= \begin{bmatrix} x \\ x-2k \end{bmatrix} F_{x-2k}, \text{ by the symmetry property of Fibonomial coefficients,}
\end{aligned}$$

from which it immediately follows that an alternating sum vanishes.

(c) Letting $x = 4m$ as identical argument shows that

$$\begin{aligned}
\begin{bmatrix} x \\ k \end{bmatrix} F_k &= \frac{F_x F_{x-1} \cdots F_{x-k+1}}{F_1 F_2 \cdots F_k} F_k, \text{ by definition of the Fibonomial coefficient,} \\
&= \frac{F_x F_{x-1} \cdots F_{x-k}}{F_1 F_2 \cdots F_{k-1}} F_{x-k+1}, \text{ by cancellation and rearrangement,} \\
&= \begin{bmatrix} x \\ k-1 \end{bmatrix} F_{x-k+1}, \text{ by definition of the Fibonomial coefficient,} \\
&= \begin{bmatrix} x \\ x-(k-1) \end{bmatrix} F_{x-(k-1)}, \text{ by the symmetry property of Fibonomial coefficients.}
\end{aligned}$$

$(-1)^{\frac{k(k+1)}{2}}$ is negative for $k \equiv 1, 2 \pmod 4$, and positive for $k \equiv 3, 4 \pmod 4$ showing that appropriate pairs cancel in the sum. This completes the proof.

Also solved by G. C. Gruebel and the proposer.

A Simple Limit

B-1041 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 45.4, November 2007)

Prove that the following expression has a limit as $n \to \infty$, and find the limit.

$$\{(F_{n+1})^{1/2} + (F_n)^{1/2}\}/(F_{n+2})^{1/2}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Since $\lim_{k\to\infty} F_k = \alpha^k/\sqrt{5}$, we find

$$\lim_{n\to\infty} \frac{(F_{n+1})^{1/2} + (F_n)^{1/2}}{(F_{n+2})^{1/2}} = \frac{\alpha^{\frac{n+1}{2}} + \alpha^{\frac{n}{2}}}{\alpha^{\frac{n+2}{2}}} = \frac{\sqrt{\alpha}+1}{\alpha}.$$

Also solved by Charles K. Cook and Rebecca Hillman (jointly), Steve Edwards, G. C. Greubel, Pentti Hankkren, Russell J. Hendel, Maitland A. Rose, H.-J. Seiffert, and the proposer.

Triangular and Fibonacci Numbers Inequality

B-1042 Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universidad de Cataluña, Barcelona, Spain (Vol. 45.4, November 2007)

Let T_n be the nth triangular number defined by $T_n = \binom{n+1}{2}$ for all $n \geq 1$. Prove that

$$\frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)^2 \geq \frac{T_{n+1}^2}{9F_nF_{n+1}}.$$

Solution by Paul S. Bruckman, Sointula, Canada

Note that $nT_{n+1}/3 = n(n+1)(n+2)/(2\cdot 3) = \binom{n+2}{3} = P_n$. Also, $\sum_{k=1}^{n} T_k = P_n$. Now

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)F_k \leq \sqrt{\sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)^2}\sqrt{\sum_{k=1}^{n}F_k^2},$$

by the Cauchy-Schwartz inequality. Also, we employ the well-known identity

$$\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}.$$

Then,

$$\sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)^2 \geq \frac{P_n^2}{F_nF_{n+1}} = \frac{n^2T_{n+1}^2}{9F_nF_{n+1}},$$

which is equivalent to the desired relation.

Also solved by Russell J. Hendel, Harris Kwong, H.-J. Seiffert, Kenneth B. Davenport, and the proposer.

Estimate for a Weighted Product

B-1043 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad de Cataluña, Barcelona, Spain (Vol. 45.4, November 2007)

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers. Prove that

$$\left(\frac{F_nF_{n+1}}{n}\right)^n \geq \prod_{cyclic} \frac{\alpha_1 F_1^2 + \cdots + \alpha_n F_n^2}{\alpha_1 + \cdots + \alpha_n} \geq (F_1F_2\cdots F_n)^2.$$

Solution by H.-J. Seiffert, Thorwaldsenstr. 13, D-12157, Berlin, Germany

First, we prove that, for all positive real numbers $x_1, x_2, \ldots, x_n$,

$$\left(\frac{1}{n}\sum_{k=1}^n x_k\right)^n \geq \prod_{k=1}^n \frac{\alpha_1 x_k + \cdots + \alpha_n x_{k+n-1}}{\alpha_1 + \cdots + \alpha_n} \geq x_1x_2\cdots x_n, \tag{1}$$

where all indices great than n are to be reduced modulo n. If $A_k = \alpha_1 x_k + \cdots + \alpha_n x_{k+n-1}$, $k = 1, 2, \ldots, n$, and $S = \alpha_1 + \cdots + \alpha_n$, then, by the weighted Arithmetic-Geometric Mean Inequality,

$$A_k/S \geq \left(x_k^{\alpha_1} x_{k+1}^{\alpha_2} \cdots x_{k+n-1}^{\alpha_n}\right)^{1/S}, \quad k = 1, 2, \ldots, n.$$

Taking the product over $k = 1, 2, \ldots, n$ gives the right-hand side inequality of (1). The Arithmetic-Geometric Mean Inequality implies

$$\prod_{k=1}^n (A_k/S) \leq \left(\frac{1}{n}\sum_{k=1}^n (A_k/S)\right)^n = \left(\frac{1}{n}\sum_{k=1}^n x_k\right)^n.$$

This proves the left-hand side inequality of (1).

To solve the present proposal, in (1) take $x_k = F_k^2$, $k = 1, \ldots, n$, and use the identity $\sum_{k=1}^n F_k^2 = F_nF_{n+1}$, which is equation (I_3) of [1].

References

[1] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, Santa Clara, CA, The Fibonacci Association, 1979.

Also solved by Paul S. Bruckman, and the proposer.