# ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

*Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.*

*If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.*

*Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2010. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".*

## BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

## PROBLEMS PROPOSED IN THIS ISSUE

### B-1056 Proposed by Charles K. Cook, Sumter, SC

If $n > 3$, show that

$$F_n^3 + L_n^3 + P_n^3 + 3F_nL_nP_n > 2(F_n + L_n)^2 P_n$$

where $P_n$ is the $n$th Pell number.

### B-1057 Proposed by Pat Costello, Eastern Kentucky University, Richmond, KY

For $n \geq 1$, prove that

$$2L_nL_{n-1} \equiv (-1)^{[\frac{n-1}{3}]}\left(5 - (-1)^{((n+2)\mathrm{mod}3)(\mathrm{mod}2)}\right) \pmod{10}$$

where $[x]$ is the greatest integer in $x$.

**B-1058 Proposed by M. N. Despande, Nagpur, India**

Prove the following identities:

(1) $9(F_{n+1}^4 + F_n^4 + F_{n-1}^4) - (F_{n+2}^4 + F_{n-2}^4) = L_n^4$;

(2) $9(L_{n+1}^4 + L_n^4 + L_{n-1}^4) - (L_{n+2}^4 + L_{n-2}^4) = 625F_n^4$.

**B-1059 Proposed by George A. Hisert, Berkeley, CA**

For any positive integer $r$, find integers $a$, $b$, $c$ and $d$ such that

$$a(L_n)^2 = b(F_{n+r})^2 + c(F_n)^2 + d(F_{n-r})^2$$

and

$$25a(F_n)^2 = b(L_{n+r})^2 + c(L_n)^2 + d(L_{n-r})^2$$

for all positive integers $n$.

**B-1060 Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Universitat Politécnica de Cataluña, Barcelona, Spain**

Let $n$ be a positive integer. Prove that

$$1 + \frac{1}{2}\left(\sum_{k=1}^{n} F_k 3^{1/F_k} + \sum_{k=1}^{n} \frac{L_k}{3^{1/L_k}}\right) > F_{2n+2}.$$

## SOLUTIONS

### A Radical Inequality

**B-1047 Proposed by Charles K. Cook, Sumter, SC (Vol. 46/47.2, May 2008/2009)**

Given positive integers $n$, $p$, and $q$ greater than 1, show that

$$\sqrt[pq]{F_n^{p+q}} + \sqrt[pq]{L_n^{p+q}} > \sqrt[p]{F_n}\sqrt[q]{L_n} + \sqrt[p]{L_n}\sqrt[q]{F_n}.$$

**Solution by H.-J. Seiffert, Thorwaldsenstr. 13, D-12157, Berlin, Germany**

For $n > 1$ and $p, q \geq 1$, let $a = \sqrt[p]{F_n}$, $b = \sqrt[q]{F_n}$, $c = \sqrt[p]{L_n}$ and $d = \sqrt[q]{L_n}$. Then, $c > a$ and $d > b$, so that $(c-a)(d-b) > 0$, or, equivalently, $ab + cd > ad + bc$. Thus, we have

$$\sqrt[pq]{F_n^{p+q}} + \sqrt[pq]{L_n^{p+q}} > \sqrt[p]{F_n}\sqrt[q]{L_n} + \sqrt[p]{L_n}\sqrt[q]{F_n},$$

which is the (corrected) desired inequality.

**Also solved by Gurdial Arora, Paul Bruckman, Harris Kwang, and the proposer.**

## A SQUEEZED FIBONACCI FRACTION

**B-1048 Proposed by José Luis Díaz-Barrero, Universidad de Cataluña, Barcelona, Spain**
**(Vol. 46/47.2, May 2008/2009)**

Let $n$ be a positive integer. Prove that

$$\tan\left(\frac{2F_{n-1}}{1+L_n^2}\right) \leq \frac{2F_{n-1}}{1+F_{2n}} \leq \tan\left(\frac{2F_{n-1}}{1+F_n^2}\right).$$

**Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY**

The given inequality becomes an equation when $n = 1$. We will show that it becomes a strict inequality when $n \geq 2$. Let $\theta_1 = \tan^{-1}(F_n)$ and $\theta_2 = \tan^{-1}(L_n)$ so that $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$. The Mean-Value Theorem asserts that

$$\frac{\tan\theta_2 - \tan\theta_2}{\theta_2 - \theta_1} = \sec^2\xi$$

for some number $\xi$ between $\theta_1$ and $\theta_2$. It follows that

$$1 + \tan^2\theta_1 = \sec^2\theta_1 < \frac{\tan\theta_2 - \tan\theta_2}{\theta_2 - \theta_1} < \sec^2\theta_2 = 1 + \tan^2\theta_2.$$

Hence,

$$\frac{\tan\theta_2 - \tan\theta_1}{1+\tan^2\theta_2} < \theta_2 - \theta_1 < \frac{\tan\theta_2 - \tan\theta_1}{1+\tan^2\theta_1},$$

which is equivalent to

$$\frac{L_n - F_n}{1+L_n^2} < \theta_2 - \theta_1 < \frac{L_n - F_n}{1+F_n^2}. \tag{1}$$

From $1 - 2F_n + F_n^2 = (1-F_n)^2 > 0$, we deduce that

$$L_n - F_n = F_{n+1} + F_{n-1} - F_n = 2F_{n-1} < 2F_n < 1 + F_n^2.$$

Therefore $(L_n - F_n)/(1+F_n^2) < 1 < \pi/2$. Consequently, (1) is also equivalent to

$$\tan\left(\frac{2F_{n-1}}{1+L_n^2}\right) < \tan(\theta_2 - \theta_1) < \tan\left(\frac{2F_{n-1}}{1+F_n^2}\right).$$

The desired result follows immediately from

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2\tan\theta_1} = \frac{L_n - F_n}{1+L_nF_n} = \frac{2F_{n-2}}{1+F_{2n}}.$$

**Also solved by Paul S. Bruckman, Kenneth Davenport, Jaroslav Seibert, H.-J. Seiffert, and the proposer.**

## Two Cubic Identities

**B-1049 Proposed by Paul S. Bruckman, Sointula, Canada**
**(Vol. 46/47.2, May 2008/2009)**

Prove the following identities:

(1) $2(L_n)^3 = -(F_{n+2})^3 + 9(F_{n+1})^3 + 9(F_{n-1})^3 + (F_{n-2})^3$;

(2) $250(F_n)^3 = -(L_{n+2})^3 + 9(L_{n+1})^3 + 9(L_{n-1})^3 + (L_{n-2})^3$.

**Solution by H.-J. Seiffert, Thorwaldsenstr. 13, D-12157, Berlin, Germany**

If $a$ and $b$ are any numbers, then

$$2(a+b)^3 = -(2a-b)^3 + 9a^3 + 9b^3 + (a-2b)^3 \tag{3}$$

as is easily verified by expanding both sides.

From the basic recurrence of the Fibonacci numbers, it follows that $2F_{n+1} - F_{n-1} = F_{n+2}$ and $F_{n+1} - 2F_{n-1} = F_{n-2}$. According to Equation ($I_8$) of [1], we have $F_{n+1} + F_{n-1} = L_n$. Now, it is easly seen that (1) follows from (3) by taking $a = F_{n+1}$ and $b = F_{n-1}$.

Similarly, since $2L_{n+1} - L_{n-1} = L_{n+2}$, $L_{n+1} - 2L_{n-1} = L_{n-2}$ and by Equation ($I_9$) of [1], $L_{n+1} + L_{n-1} = 5F_n$, the desired identity (2) follows from (3) by setting $a = L_{n+1}$ and $b = L_{n-1}$.

## References

[1] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, Santa Clara, CA, The Fibonacci Association, 1979.

**Also solved by Charles Cook, Kenneth Davenport, G. C. Greubel, Harris Kwang, David Mitchell, John Morrison, Jaroslav Seibert, James A. Sellers, and the proposer.**

## A Recurrence Relation

**B-1050 Proposed by R. David Mitchell, University of South Carolina Sumter, Sumter, SC**
**(Vol. 46/47.2, May 2008/2009)**

For the generalized Fibonacci sequence $ja_{n-1} + ka_n = a_{n+1}, n \geq 2$, where $j$, $k$, $a_1$, and $a_2$ are non-zero integers, find functions $f(j,k,a_1,a_2)$ and $g(j,n)$ such that $a_n^2 - a_{n-1}a_{n+1} = f(j,k,a_1,a_2)g(j,n)$.

**Solution by Paul S. Bruckman, Surrey, British Columbia, Canada**

By induction, the following matrix relation holds:

$$\begin{pmatrix} k & j \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a_2 & a_1 \\ a_1 & a_0 \end{pmatrix} = \begin{pmatrix} a_{n+2} & a_{n+1} \\ a_{n+1} & a_n \end{pmatrix}. \tag{*}$$

In (*), we find by backward application of the given recurrence that $a_0 = \{a_2 - ka_1\}/j$. Taking determinants in (*), we obtain: $(-j)^n\{a_2a_0 - (a_1)^2\} = \{a_{n+2}a_n - (a_{n+1})^2\}$. Letting $w_n = (a_n)^2 - a_{n+1}a_{n-1}$, this becomes $w_n = w_1(-j)^{n-1}$, $n = 1, 2, \ldots$. Now $w_1 = a_2a_0 - (a_1)^2 = \{(a_2)^2 - ka_1a_2\}/j - (a_1)^2 = \{(a_2)^2 - ka_1a_2 - j(a_1)^2\}/j$. Therefore, $w_n = \{j(a_1)^2 + ka_1a_2 - (a_2)^2\}(-j)^{n-2}$; we may then take $f = j(a_1)^2 + ka_1a_2 - (a_2)^2$ and $g = (-j)^{n-2}$.

All solvers basically gave the same solution. H.-J. Seiffert pointed out that the equation $a_n^2 - a_{n-1}a_{n+1} = [a_1^2 + \frac{1}{j}(ka_1 - a_2)a_2](-j)^n$ appeared in S. Rabinowitz's paper, *Algorithmic Manipulation of Second-Order Linear Recurrences*, The Fibonacci Quarterly, **37.2** (1999), 162–177.

**Also solved by G. C. Greubel, Harris Kwang, Jaroslav Seibert, H.-J. Seiffert, and the proposer.**

*We wish to acknowledge Angela Plaza and Sergio Falcon for their joint solutions to problems B-1044 and B-1045.*