ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
HARRIS KWONG

Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2022. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1301 Proposed by Peter Ferraro, Roselle Park, NJ.

Show that, for any integer $n \geq 0$,

$$\left\lfloor \sqrt{F_{2n+1}F_{2n+2}L_{2n+3}} \right\rfloor = F_{3n+3}.$$

B-1302 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Evaluate

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_nF_{n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(F_nF_{n+1})^2}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_nF_{n+3}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(F_nF_{n+3})^2}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^n}{(F_nF_{n+3})^2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(F_nF_{n+1})^2}$

B-1303 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

Prove that $\left\lfloor x+\frac{1}{x}\right\rfloor = \left\lfloor x^2+\frac{1}{x^2}\right\rfloor$ if and only if $\frac{1}{\alpha} < x < \alpha$.

B-1304 **Proposed by Toyesh Prakash Sharma (high school student), St. C. F. Andrews School, Agra, India.**

Let n be a positive integer. Show that

$$\frac{F_n^2}{\ln\left(1+F_n^2\right)} + \frac{F_{n+1}^2}{\ln\left(1+F_{n+1}^2\right)} + \frac{F_{n+2}^2}{\ln\left(1+F_{n+2}^2\right)} > F_{n+2}.$$

B-1305 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For any positive integer n, prove that

$$\sum_{k=1}^{n} F_{F_{3k}} F_{L_{3k}} L_{F_{3k}} = F_{F_{3n+1}} F_{F_{3n+2}} - 1.$$

SOLUTIONS

The Seventh and Last Oldie from the Vault

B-51 **Proposed by Douglas Lind, Falls Church, VA.**
(Vol. 2.3, October 1964)

Let $\phi(n)$ be the Euler totient and let $\phi^k(n)$ be defined by $\phi^1(n) = \phi(n)$, $\phi^{k+1}(n) = \phi(\phi^k(n))$. Prove that $\phi^n(F_n) = 1$, where F_n is the nth Fibonacci number.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We note that given any positive integer m, $\phi(m)$ is odd if and only if $m = 1$ or $m = 2$, in which case $\phi(m) = 1$. Also, $\phi(m) \le m/2$ if m is even. We observe as well that for each positive integer n, $F_n \le 2^{n-1}$, with equality if and only if $n = 1$.

For $n \in \{1, 2, 3\}$, we have $\phi(F_n) = 1$ and thus $\phi^n(F_n) = 1$. For $n \ge 3$, $\phi(F_n)$ is even, so $\phi^2(F_n) \le \phi(F_n)/2$. Continuing inductively, either $\phi^k(F_n) = 1$ for some $k \in \{2, 3, \ldots, n\}$, or

$$\phi^n(F_n) \le \frac{\phi^{n-1}(F_n)}{2} \le \frac{\phi^{n-2}(F_n)}{2^2} \le \cdots \le \frac{\phi(F_n)}{2^{n-1}} < \frac{F_n}{2^{n-1}} \le 1,$$

which is impossible. Hence $\phi^k(F_n) = 1$ for some $k \in \{2, 3, \ldots, n\}$, so $\phi^n(F_n) = 1$.

Addendum. For each positive integer n, let $g(n)$ denote the minimum value of k for which $\phi^k(F_n) = 1$. We observe that g is non-decreasing for $1 \le n \le 11$, but $g(11) > g(12)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$g(n)$	1	1	1	2	3	3	4	4	5	6	7	6	8	8	8	9	9	10	11	12

Other interesting values of $g(n)$ for small n include $g(40) = g(42) = 23$ and $g(41) = 26$.

Questions:

1. What is the range of $g(n)$? For example, it appears that neither 14 nor 24 is in the range.
2. What is the asymptotic behavior of $g(n)$? Is $g(n)$ asymptotic to $c\log_2(F_n)$ for some constant c as n approaches infinity?
3. What is the limit of $g(n)/n$ as n approaches infinity? Does it equal $c\log_2(\alpha)$?
4. Can we determine an explicit formula for $g(n)$?

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We use a theorem by Pillai [1] stating that if $R(n)$ is the least integer r for which the rth iterate of the Euler totient function evaluated at n equals 1, then

$$R(n) \le \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1.$$

Clearly, $R(F_1) = R(1) = 1$. If $n \ge 2$, then by Pillai's theorem and that $F_n \le 2^{n-1}$,

$$R(F_n) \le \left\lfloor \frac{\log F_n}{\log 2} \right\rfloor + 1 \le \left\lfloor \frac{\log 2^{n-1}}{\log 2} \right\rfloor + 1 = n,$$

and we are done.

Editor's Note: Fedak observed that if $\frac{m}{2^{n-1}} \le 2$, then $\phi^n(m) = 1$. This leads to the desired result immediately because $F_n < 2^n$. Park proved that if $\phi^{k-2}(a) \ge 3$, then $\phi^k(a) \le \frac{1}{2^{k-1}}a$, which completes the proof because $F_n \le 2^{n-1}$.

References

[1] S. S. Pillai, *On a function connected with $\varphi(n)$*, Bull. Amer. Math. Soc., **35** (1929), 837–841.

Also solved by Saunak Bhattacharjee (undergraduate), I. V. Fedak, Heng Gao, Ho Park, Raphael Schumacher (graduate student), and David Terr.

A Consequence of the Minkowski Inequality

B-1281 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 59.1, February 2021)

For all positive integers n and m, prove that

$$L_1 + \sqrt[n]{\sum_{k=1}^{m} L_k^n} \le \sqrt[n]{\sum_{k=1}^{m} F_{k+1}^n} + F_{m+1}.$$

Solution by Ángel Plaza, Universidad de Las Palmos de Gran Canaria, Gran Canaria, Spain.

Because $L_1 = 1$, $L_k = F_{k-1} + F_{k+1}$, and $\sum_{k=1}^{m-1} F_k = F_{m+1} - 1$, the proposed inequality may be written equivalently as

$$\sqrt[n]{\sum_{k=1}^{m}\left(F_{k-1}+F_{k+1}\right)^n} \le \sqrt[n]{\sum_{k=1}^{m}F_{k+1}^n}+\sum_{k=1}^{m}F_{k-1}.$$

Now, because $\sqrt[n]{\sum_{k=1}^{m}F_{k-1}^n} \le \sum_{k=1}^{m}F_{k-1}$, it is enough to prove that

$$\sqrt[n]{\sum_{k=1}^{m}(F_{k-1}+F_{k+1})^n} \le \sqrt[n]{\sum_{k=1}^{m}F_{k+1}^n}+\sqrt[n]{\sum_{k=1}^{m}F_{k-1}^n},$$

which is the Minkowski inequality.

Also solved by Dmitry Fleischman, Haydn Gwyn (undergraduate), Hideyuki Ohtsuka, Albert Stadler, and the proposer.

Both Summations Are Telescopic

B-1282 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 59.1, February 2021)

For any positive integer n, find closed form expressions for the sums

$$\sum_{k=1}^{n}F_{3k}F_{3k+1}, \qquad \text{and} \qquad \sum_{k=1}^{n}F_{2F_{3k}}F_{2F_{3k+1}}.$$

Solution by Raphael Schumacher (graduate student), ETH Zurich, Switzerland.

We have for all $k \in \mathbb{N}$ the identities

$$\frac{F_{3k+2}^2-F_{3k-1}^2}{4}=\frac{\left(F_{3k+1}+F_{3k}\right)^2-\left(F_{3k+1}-F_{3k}\right)^2}{4}=F_{3k}F_{3k+1},$$

$$F_{2F_{3k}}F_{2F_{3k+1}}=F_{F_{3k+1}+F_{3k}}^2-F_{F_{3k+1}-F_{3k}}^2=F_{F_{3k+2}}^2-F_{F_{3k-1}}^2.$$

In the second identity, we have used for $a \ge b$ the identity

$$F_{2a}F_{2b}=F_{a+b}^2-F_{a-b}^2,$$

which can be derived from the Binet's formula, the Catalan's identity, or the product formula $5F_sF_t=L_{s+t}-(-1)^tL_{s-t}$. By telescoping, it follows from the above two identities with $F_2=1$ that

$$\sum_{k=1}^{n}F_{3k}F_{3k+1}=\sum_{k=1}^{n}\frac{F_{3k+2}^2-F_{3k-1}^2}{4}=\frac{F_{3n+2}^2-1}{4},$$

$$\sum_{k=1}^{n}F_{2F_{3k}}F_{2F_{3k+1}}=\sum_{k=1}^{n}\left(F_{F_{3k+2}}^2-F_{F_{3k-1}}^2\right)=F_{F_{3n+2}}^2-1.$$

Editor's Note: Several solvers used the Binet's formulas to set up a summation of arithmetic progressions in α and β, thereby obtaining $\sum_{k=1}^{n}F_{3k}F_{3k+1}=\frac{1}{20}\left(L_{6n+4}-5-2(-1)^n\right)$. Gwyn, using a slightly different set up, showed that $\sum_{k=1}^{n}F_{3k}F_{3k+1}=\frac{1}{4}\left(2F_{3n+2}F_{3n+3}-F_{6n+4}-1\right)$. Meanwhile, many solvers used the identity $F_{2F_{3k}}F_{2F_{3k+1}}=\frac{1}{5}\left(L_{2F_{3k+2}}-L_{2F_{3k-1}}\right)$ to derive the result $\sum_{k=1}^{n}F_{2F_{3k}}F_{2F_{3k+1}}=\frac{1}{5}\left(L_{2F_{3n+2}}-3\right)$.

Also solved by Michel Bataille, Brian Bradie, Steve Edwards, I. V. Fedak, Robert Frontczak, G. C. Greubel, Haydn Gwyn (undergraduate), Ángel Plaza, Albert Stadler, David Terr, Dan Weiner, and the proposer. Thomas Achammer and Dmitry Fleischman, independently, only found the closed form expression for the first summation.

Rewriting the Summation Does the Magic

B-1283 Proposed by Michel Bataille, Rouen, France.
(Vol. 59.1, February 2021)

For positive integers m, n, evaluate in closed form:

$$\sum_{j=1}^{n} \binom{2n}{n-j} \frac{F_{mn-4j} + F_{mn+4j}}{F_{mn}}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Note that

$$\begin{aligned}
\sum_{j=1}^{n} \binom{2n}{n-j} \frac{F_{mn-4j} + F_{mn+4j}}{F_{mn}} &= \frac{1}{F_{mn}} \left[\sum_{j=1}^{n} \binom{2n}{n-j} F_{mn-4j} + \sum_{j=1}^{n} \binom{2n}{n+j} F_{mn+4j} \right] \\
&= \frac{1}{F_{mn}} \left[\sum_{i=0}^{n-1} \binom{2n}{i} F_{mn-4n+4i} + \sum_{i=n+1}^{2n} \binom{2n}{i} F_{mn-4n+4i} \right] \\
&= \frac{1}{F_{mn}} \sum_{i=0}^{2n} \binom{2n}{i} F_{mn-4n+4i} - \binom{2n}{n}.
\end{aligned}$$

Now, by the binomial theorem,

$$\begin{aligned}
\sum_{i=0}^{2n} \binom{2n}{i} F_{mn-4n+4i} &= \frac{1}{\sqrt{5}} \sum_{i=0}^{2n} \binom{2n}{i} \left(\alpha^{mn-4n+4i} - \beta^{mn-4n+4i} \right) \\
&= \frac{1}{\sqrt{5}} \left[\alpha^{mn-4n} (1+\alpha^4)^{2n} - \beta^{mn-4n} (1+\beta^4)^{2n} \right] \\
&= \frac{1}{\sqrt{5}} \left[\alpha^{mn} (\alpha^2+\beta^2)^{2n} - \beta^{mn} (\alpha^2+\beta^2)^{2n} \right] \\
&= L_2^{2n} F_{mn}.
\end{aligned}$$

Hence,

$$\sum_{j=1}^{n} \binom{2n}{n-j} \frac{F_{mn-4j} + F_{mn+4j}}{F_{mn}} = 9^n - \binom{2n}{n}.$$

Moreover, if $G_{k,n}$ denotes the nth generalized k-Fibonacci number defined by $G_{k,n+1} = kG_{k,n} + G_{k,n-1}$, with $G_{k,0} = a$ and $G_{k,1} = b$, then the expression $\alpha^2 + \beta^2$ becomes $\alpha_k^2 + \beta_k^2$, where

$$\alpha_k = \frac{k + \sqrt{k^2+4}}{2} \qquad \text{and} \qquad \beta_k = \frac{k - \sqrt{k^2+4}}{2}.$$

Thus, $\alpha_k^2 + \beta_k^2 = k^2 + 2$, and

$$\sum_{j=1}^{n} \binom{2n}{n-j} \frac{G_{k,mn-4j} + G_{k,mn+4j}}{G_{k,mn}} = (k^2+2)^{2n} - \binom{2n}{n}.$$

Editor's Note: Greubel obtained the following generalization for any integer $q \geq 0$:

$$\sum_{j=1}^{n} \binom{2n}{n-j} \frac{F_{mn-2pj} + F_{mn+2pj}}{F_{mn}} = \begin{cases} L_{2q}^{2n} - \binom{2n}{n} & \text{if } p = 2q, \\ 5^n F_{2q+1}^{2n} - \binom{2n}{n} & \text{if } p = 2q+1; \end{cases}$$

and Tuenter showed that, for any integers a, b and n, where $n \geq 0$,

$$\sum_{j=0}^{n} \binom{2n}{n-j} \frac{F_{b-4aj} + F_{b+4aj}}{F_b} = L_{2a}^{2n} + \binom{2n}{n}.$$

Also solved by I. V. Fedak, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Haydn Gwyn (undergraduate), Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, David Terr, Hans J. H. Tuenter, and the proposer.

A Logarithmic Inequality

B-1284 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Gran Canaria, Spain.
(Vol. 59.1, February 2021)

Let $(x_n)_{n\geq 0}$ be the sequence recurrently defined by $x_{n+1} = x_n + x_{n-1}$ for $n \geq 1$, with initial conditions $x_0 \geq 0$ and $x_1 \geq 1$. For $n \geq 2$, prove that

$$\ln\left(\frac{1}{n-1}\left(\frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}}\right)\right) \geq \frac{2}{n-1}\left(\frac{x_0}{x_3} + \frac{x_1}{x_4} + \cdots + \frac{x_{n-2}}{x_{n+1}}\right).$$

Solution 1 by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let $f(t) = \ln t - \frac{2(t-1)}{t+1}$. Then, $f(1) = 0$ and $f'(t) = \frac{1}{t} - \frac{4}{(t+1)^2} = \frac{(t-1)^2}{t(t+1)^2} \geq 0$ for all $t \geq 1$. Therefore, $\ln t \geq \frac{2(t-1)}{t+1}$ for all $t \geq 1$. Using $x_0 \geq 0$ and $x_1 > 0$, we gather that $x_{k+1}/x_k \geq 1$ for all positive integers k. Thus, for $n \geq 2$,

$$\begin{aligned} \ln\left(\frac{1}{n-1}\left(\frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}}\right)\right) &\geq \ln\left(\frac{x_2}{x_1} \cdot \frac{x_3}{x_2} \cdot \cdots \cdot \frac{x_n}{x_{n-1}}\right)^{\frac{1}{n-1}} \\ &= \frac{1}{n-1}\left(\ln\frac{x_2}{x_1} + \ln\frac{x_3}{x_2} + \cdots + \ln\frac{x_n}{x_{n-1}}\right) \\ &\geq \frac{2}{n-1}\left(\frac{\frac{x_2}{x_1} - 1}{\frac{x_2}{x_1} + 1} + \frac{\frac{x_3}{x_2} - 1}{\frac{x_3}{x_2} + 1} + \cdots + \frac{\frac{x_n}{x_{n-1}} - 1}{\frac{x_n}{x_{n-1}} + 1}\right) \\ &= \frac{2}{n-1}\left(\frac{x_0}{x_3} + \frac{x_1}{x_4} + \cdots + \frac{x_{n-2}}{x_{n+1}}\right). \end{aligned}$$

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

For $s > t > 0$, the log mean inequality asserts that

$$\frac{s-t}{\ln s - \ln t} \le \frac{s+t}{2},$$

that is,

$$\frac{2(s-t)}{s+t} \le \ln s - \ln t.$$

This inequality also holds for $s = t > 0$. Putting $s = x_{k+1}$ and $t = x_k$ for $k \ge 1$ in the above inequality, we find

$$\frac{2x_{k-1}}{x_{k+2}} \le \ln x_{k+1} - \ln x_k.$$

Therefore,

$$\sum_{k=1}^{n-1} \frac{2x_{k-1}}{x_{k+2}} \le \sum_{k=1}^{n-1} \big(\ln x_{k+1} - \ln x_k\big) = \ln x_n - \ln x_1 = \ln \frac{x_n}{x_1}.$$

Using the AM-GM inequality, we have

$$(n-1)\ \ln\left(\frac{1}{n-1}\sum_{k=1}^{n-1}\frac{x_{k+1}}{x_k}\right) \ge (n-1)\ln\left(\prod_{k=1}^{n-1}\frac{x_{k+1}}{x_k}\right)^{\frac{1}{n-1}} = \ln\frac{x_n}{x_1}.$$

Combining the last two results, we obtain the desired inequality.

Also solved by Michel Bataille, Dmitry Fleischman, Haydn Gwyn (undergraduate), Albert Stadler, and the proposer.

It Is the Binomial Theorem!

B-1285 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 59.1, February 2021)

Let $i = \sqrt{-1}$. For any integer $n \ge 0$, prove that

(i) $\displaystyle\sum_{k=-n}^{n} \binom{2n}{n+k}\left(e^{\frac{2k\pi i}{5}} + e^{\frac{4k\pi i}{5}}\right) = L_{2n};$

(ii) $\displaystyle\sum_{k=-n}^{n} \binom{2n}{n+k}\left(e^{\frac{k\pi i}{5}} + (-1)^n e^{\frac{3k\pi i}{5}}\right) = \big(\sqrt{5}\,\big)^n L_n.$

Solution by Michel Bataille, Rouen, France.

We first derive a lemma: for $\theta \in \mathbb{R}$, we have

$$\sum_{k=-n}^{n}\binom{2n}{n+k}e^{ik\theta} = e^{-in\theta}\sum_{j=0}^{2n}\binom{2n}{j}e^{ij\theta}$$
$$= e^{-in\theta}(1+e^{i\theta})^{2n} = \left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\right)^{2n} = \left(2\cos\frac{\theta}{2}\right)^{2n}$$

(i) Because $2\cos\frac{\pi}{5}=\alpha$, and $2\cos\frac{2\pi}{5}=-\beta$, the lemma gives

$$\sum_{k=-n}^{n}\binom{2n}{n+k}\left(e^{\frac{2k\pi i}{5}}+e^{\frac{4k\pi i}{5}}\right)=\left(2\cos\frac{\pi}{5}\right)^{2n}+\left(2\cos\frac{2\pi}{5}\right)^{2n}=\alpha^{2n}+(-\beta)^{2n}=L_{2n}.$$

(ii) From

$$4\cos^2\frac{\pi}{10}=2+2\cos\frac{\pi}{5}=\alpha+2=\alpha^2+1=\alpha^2-\alpha\beta,$$

and

$$-4\cos^2\frac{3\pi}{10}=-2-2\cos\frac{3\pi}{5}=-2+2\cos\frac{2\pi}{5}=-2-\beta=\alpha\beta-\beta^2,$$

we obtain, after applying the lemma again,

$$\begin{aligned}\sum_{k=-n}^{n}\binom{2n}{n+k}\left(e^{\frac{k\pi i}{5}}+(-1)^n e^{\frac{3k\pi i}{5}}\right) &= \left(2\cos\frac{\pi}{10}\right)^{2n}+(-1)^n\left(2\cos\frac{3\pi}{10}\right)^{2n}\\ &= \left(4\cos^2\frac{\pi}{10}\right)^{n}+\left(-4\cos^2\frac{3\pi}{10}\right)^{n}\\ &= (\alpha^2-\alpha\beta)^n+(\alpha\beta-\beta^2)^n\\ &= (\alpha-\beta)^n(\alpha^n+\beta^n)\\ &= \left(\sqrt{5}\right)^n L_n.\end{aligned}$$

Also solved by Brian Bradie, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Haydn Gwyn (undergraduate), Dipyaman Mukherjee (high school student), Ángel Plaza, Albert Stadler, David Terr, and the proposer.

Correction. In Problem B-1299, the name of the school should be St. C. F. Andrews School.