ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2023. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1321 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $A = \{(n,r) \mid n, r \in \mathbb{N}$ with $n \geq 3$, $r \geq 1$, and, if n is even, then r is even$\}$. Prove that

$$\sum_{(n,r)\in A} \frac{1}{F_n^r} = \frac{61 - 15\sqrt{5}}{18}.$$

B-1322 Proposed by Mihaly Bencze, Braşov, Romania, and Neculai Stanciu, Buzău, Romania.

Prove that $\displaystyle\frac{(n-1)^2}{n}\sum_{k=1}^{n}\left(\frac{F_k}{F_{n+2}-F_k-1}\right)^2 \geq 1$ for any integer $n > 1$.

B-1323 Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Arga, India.

Let n be a positive integer. Show that ${F_n}^{F_n} + {L_n}^{L_n} \geq 2{F_{n+1}}^{F_{n+1}}$.

B-1324 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Let $\{x\}$ denote the fractional part of the real number x. Evaluate $\lim_{n\to\infty}\left\{\alpha F_{2n}^2\right\}$ and $\lim_{n\to\infty}\left\{\alpha F_{2n-1}^2\right\}$.

B-1325 **Proposed by Hans J. H. Tuenter, Toronto, Canada.**

Let $\{\mu_n\}$ be a sequence that follows the recurrence relation $\mu_{n+3} = \mu_{n+2} + \mu_{n+1} + \mu_n$, with arbitrary initial values μ_0, μ_1, and μ_2. Prove that, for such a generalized Tribonacci sequence, the sum of eight consecutive numbers always equals four times the seventh of these numbers.

SOLUTIONS

The Product of Three Consecutive Fibonacci and Lucas Numbers

B-1301 **Proposed by Peter Ferraro, Roselle Park, NJ.**
(Vol. 60.1, February 2022)

Show that, for any integer $n \geq 0$,

$$\left\lfloor \sqrt{F_{2n+1}F_{2n+2}L_{2n+3}} \right\rfloor = F_{3n+3}.$$

Solution by Michael Bacon and Charles Cook (jointly), Sumter, SC.

It suffices to prove that

$$F_{3n+3}^2 \leq F_{2n+1}F_{2n+2}L_{2n+3} < (F_{3n+3}+1)^2.$$

To prove the above, we use the following identities

$$\begin{aligned} F_m &= \frac{L_{m+1}+L_{m-1}}{5}, \\ L_mL_k &= L_{m+k} + (-1)^k L_{m-k}, \\ F_mF_k &= \frac{L_{m+k} - (-1)^k L_{m-k}}{5}, \\ F_mL_k &= F_{m+k} + (-1)^k F_{m-k}. \end{aligned}$$

They lead to

$$F_{2n+1}F_{2n+2} = \frac{L_{4n+3}+1}{5},$$

and

$$(L_{4n+3}+1)L_{2n+3} = L_{6n+6} - L_{2n} + L_{2n+3} = L_{6n+6} + 2L_{2n+1}.$$

So

$$F_{3n+3}^2 = \frac{L_{6n+6} - 2(-1)^{3k+3}}{5} \leq \frac{L_{6n+6}+2L_{2n+1}}{5} = \frac{(L_{4n+3}+1)L_{2n+3}}{5} = F_{2n+1}F_{2n+2}L_{2n+3},$$

which shows the first inequality. Next,

$$\begin{aligned}
F_{2n+1}F_{2n+2}L_{2n+3} &= \frac{L_{6n+6}+2L_{2n+1}}{5}\\
&= \frac{L_{6n+6}-2(-1)^{3n+3}}{5}+\frac{2L_{2n+1}}{5}+\frac{2(-1)^{3n+3}}{5}\\
&< \frac{L_{6n+6}-2(-1)^{3n+3}}{5}+\frac{2(L_{3n+4}+L_{3n+2})}{5}+1\\
&= F_{3n+3}^2+2F_{3n+3}+1\\
&= (F_{3n+3}+1)^2,
\end{aligned}$$

which proves the second inequality.

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Brian Bradie, High School Summer Research Group at the Citadel (Ethan Curb, Peyton Matheson, Aiden Milligan, Cameron Moening, Virginia Rhett Smith and Ell Torek), Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.

Three Infinite Series of Reciprocals of Products of Fibonacci Numbers

B-1302 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 60.1, February 2022)

Evaluate

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+1})^2}$

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+3}}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+3})^2}$

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{2(-1)^n}{(F_nF_{n+3})^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+1})^2}$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

(i) Note that both series are absolutely convergent, therefore their sum is equal to

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+1}}+\frac{(-1)^n}{(F_nF_{n+1})^2}\right)=\sum_{n=1}^{\infty}\frac{F_nF_{n+1}+(-1)^n}{(F_nF_{n+1})^2}.$$

Because of d'Ocagne's identity, we have

$$F_nF_{n+1}+(-1)^n=F_{n+2}F_{n-1}=(F_{n+1}+F_n)(F_{n+1}-F_n)=F_{n+1}^2-F_n^2.$$

The last series telescopes, and its sum is

$$\sum_{n=1}^{\infty}\frac{F_nF_{n+1}+(-1)^n}{(F_nF_{n+1})^2}=\sum_{n=1}^{\infty}\left(\frac{1}{F_n^2}-\frac{1}{F_{n+1}^2}\right)=1.$$

(ii) As in (i), the sum is equal to

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+3}}+\frac{(-1)^n}{(F_nF_{n+3})^2}\right)=\sum_{n=1}^{\infty}\frac{F_nF_{n+3}+(-1)^n}{(F_nF_{n+3})^2}.$$

Again, d'Ocagne's identity implies that

$$F_nF_{n+3}+(-1)^n=F_{n+2}F_{n+1}=\frac{(F_{n+2}+F_{n+1})^2-(F_{n+2}-F_{n+1})^2}{4}=\frac{F_{n+3}^2-F_n^2}{4}.$$

The last series telescopes, and its sum is

$$\sum_{n=1}^{\infty}\frac{F_nF_{n+3}+(-1)^n}{(F_nF_{n+3})^2}=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{F_n^2}-\frac{1}{F_{n+3}^2}\right)=\frac{1}{4}\left(1+1+\frac{1}{4}\right)=\frac{9}{16}.$$

(iii) Let the sums in the three problems be A, B, and C, respectively. Taking into account the absolute convergence of the involved series, we find

$$C=2B-A+\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+1}}-\frac{2}{F_nF_{n+3}}\right).$$

Because $F_{n+3}-2F_{n+1}=F_n$,

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+1}}-\frac{2}{F_nF_{n+3}}\right)=\sum_{n=1}^{\infty}\frac{1}{F_{n+1}F_{n+3}}=\sum_{n=1}^{\infty}\left(\frac{1}{F_{n+1}F_{n+2}}-\frac{1}{F_{n+2}F_{n+3}}\right)=\frac{1}{2}.$$

Therefore,

$$C=2\cdot\frac{9}{16}-1+\frac{1}{2}=\frac{5}{8}.$$

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Charles Cook, Dmitry Fleischman, Robert Frontczak, Albert Stadler, Andrés Ventas, and the proposer.

When Do We Reach the Same Floor?

B-1303 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 60.1, February 2022)

Prove that $\left\lfloor x+\frac{1}{x}\right\rfloor=\left\lfloor x^2+\frac{1}{x^2}\right\rfloor$ if and only if $\frac{1}{\alpha}<x<\alpha$.

Solution 1 by Hideyuki Ohtsuka, Saitama, Japan.

We have

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2. \tag{1}$$

(i) Assume $\frac{1}{\alpha}<x<\alpha$. Then,

$$0>\frac{1}{x}(x-\alpha)\left(x-\frac{1}{\alpha}\right)=x+\frac{1}{x}-\alpha-\frac{1}{\alpha}=x+\frac{1}{x}-\sqrt{5}. \tag{2}$$

By (2) and the AM-GM inequality, we have

$$2\le x+\frac{1}{x}<\sqrt{5}.$$

By the above inequality and (1), we have

$$2 \leq x^2 + \frac{1}{x^2} < 3.$$

Therefore, we obtain

$$\left\lfloor x + \frac{1}{x} \right\rfloor = \left\lfloor x^2 + \frac{1}{x^2} \right\rfloor = 2.$$

(ii) Next, assume

$$\left\lfloor x + \frac{1}{x} \right\rfloor = \left\lfloor x^2 + \frac{1}{x^2} \right\rfloor = n.$$

Then,

(a) $n \leq x + \frac{1}{x} < n + 1$, and
(b) $n \leq x^2 + \frac{1}{x^2} < n + 1$.

Note that $n \geq 2$ and $x > 0$ because $x^2 + \frac{1}{x^2} \geq 2$. By (1) and (b), we have

$$n \leq \left(x + \frac{1}{x}\right)^2 - 2 < n + 1.$$

That is,

$$\sqrt{n+2} \leq x + \frac{1}{x} < \sqrt{n+3}. \tag{3}$$

For $n \geq 3$, we have $\sqrt{n+3} < n$, because

$$n^2 - \left(\sqrt{n+3}\right)^2 = n(n-1) - 3 > 0.$$

Thus, (a) does not hold when $n \geq 3$. Hence, $n = 2$, and (3) becomes

$$2 \leq x + \frac{1}{x} < \sqrt{5}.$$

The solution to the second inequality is

$$\frac{1}{\alpha} = \frac{\sqrt{5}-1}{2} < x < \frac{\sqrt{5}+1}{2} = \alpha.$$

Therefore, by (i) and (ii), we conclude that $\lfloor x + \frac{1}{x} \rfloor = \lfloor x^2 + \frac{1}{x^2} \rfloor$ if and only if $\frac{1}{\alpha} < x < \alpha$.

Solution 2 by the proposer.

From the inequality $x^2 + \frac{1}{x^2} \geq 2$, it follows that $x + \frac{1}{x} \geq 2$, because $\left(x + \frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2} \geq 2$. Hence, $x > 0$. If $y = x + \frac{1}{x} \geq 3$, then

$$\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) = y^2 - 2 - y = (y+1)(y-2) \geq 4 \cdot 1 > 1.$$

Therefore, if $\lfloor x + \frac{1}{x} \rfloor = \lfloor x^2 + \frac{1}{x^2} \rfloor$, we must have

$$\left\lfloor x + \frac{1}{x} \right\rfloor = \left\lfloor x^2 + \frac{1}{x^2} \right\rfloor = 2.$$

From the inequality $x + \frac{1}{x} < 3$, by $x > 0$ we obtain $x^2 - 3x + 1 < 0$, and

$$\beta^2 = \frac{3-\sqrt{5}}{2} < x < \frac{3+\sqrt{5}}{2} = \alpha^2.$$

Similarly, from the inequality $x^2 + \frac{1}{x^2} < 3$, by $x > 0$ we obtain

$$-\beta = \frac{\sqrt{5}-1}{2} = \sqrt{\frac{3-\sqrt{5}}{2}} < x < \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{\sqrt{5}+1}{2} = \alpha.$$

Finally, using

$$\beta^2 < -\beta = \frac{1}{\alpha} < \alpha < \alpha^2,$$

we deduce that $\frac{1}{\alpha} < x < \alpha$.

Also solved by Ulrich Abel, Thomas Achammer, Michael Bacon and Charles Cook (jointly), Michel Bataille, Brian Bradie, Mei-Qin Chen and Lavender Milligan (high school students), Dmitry Fleischman, Sungwoo Hwang, Ángel Plaza, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, Seán M. Stewart, David Terr, and Andrés Ventas.

An Easy Inequality?

B-1304 Proposed by Toyesh Prakash Sharma (student), St. C. F. Andrews School, Agra, India.
(Vol. 60.1, February 2022)

Let n be a positive integer. Show that

$$\frac{F_n^2}{\ln\left(1+F_n^2\right)} + \frac{F_{n+1}^2}{\ln\left(1+F_{n+1}^2\right)} + \frac{F_{n+2}^2}{\ln\left(1+F_{n+2}^2\right)} > F_{n+2}.$$

Solution 1 by Hideyuki Ohtsuka, Saitama, Japan.

Let $x > 0$. Because

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} > 1 + 2x + 2x^2 > 1 + x^2,$$

we have $2x > \ln(1+x^2)$, from which we find $\frac{x^2}{\ln(1+x^2)} > \frac{x}{2}$. Therefore, we obtain

$$\frac{F_n^2}{\ln\left(1+F_n^2\right)} + \frac{F_{n+1}^2}{\ln\left(1+F_{n+1}^2\right)} + \frac{F_{n+2}^2}{\ln\left(1+F_{n+2}^2\right)} > \frac{F_n + F_{n+1} + F_{n+2}}{2} = F_{n+2}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

Because every term on the left side of the given inequality is positive, it is enough to show that $\frac{F_{n+2}^2}{\ln\left(1+F_{n+2}^2\right)} > F_{n+2}$. Let $x > 0$. We note that

$$\frac{x^2}{\ln(1+x^2)} > x, \tag{4}$$

because it is equivalent to $e^x > 1 + x^2$, which follows from

$$e^x - (1+x^2) > 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - (1+x^2) = x - \frac{x^2}{2} + \frac{x^3}{6} = x\left(1 - \frac{x}{4}\right)^2 + \frac{5x^3}{48} > 0.$$

Editor's Note: Using (4), Frontczak derived the stronger result

$$\frac{F_n^2}{\ln\left(1+F_n^2\right)}+\frac{F_{n+1}^2}{\ln\left(1+F_{n+1}^2\right)}+\frac{F_{n+2}^2}{\ln\left(1+F_{n+2}^2\right)}>2F_{n+2}.$$

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Dmitry Fleischman, Robert Frontczak, Wei-Kai Lai and John Risher (jointly), Ángel Plaza, Aiden Milligan (high school student), Andrés Ventas, and the proposer.

Product of Fibnoacci/Lucas Numbers With Fibonacci/Lucas Subscripts

B-1305 Proposed by Hideyuki Ohtsuka, Saitama Japan. (Vol. 60.1, February 2022)

For any positive integer n, prove that

$$\sum_{k=1}^{n} F_{F_{3k}}F_{L_{3k}}L_{F_{3k}}=F_{F_{3n+1}}F_{F_{3n+2}}-1.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Using the identities $F_nL_n=F_{2n}$ and $F_mF_n=\frac{1}{5}\left(L_{m+n}-(-1)^nL_{m-n}\right)$, it follows that

$$F_{F_{3k}}F_{L_{3k}}L_{F_{3k}}=F_{2F_{3k}}F_{L_{3k}}=\frac{1}{5}\left(L_{2F_{3k}+L_{3k}}-(-1)^{2F_{3k}}L_{L_{3k}-2F_{3k}}\right).$$

Now,

$$2F_{3k}+L_{3k}=2F_{3k}+F_{3k-1}+F_{3k+1}=F_{3k+3}.$$
$$L_{3k}-2F_{3k}=F_{3k-1}+F_{3k+1}-2F_{3k}=F_{3k-3},$$

so

$$F_{F_{3k}}F_{L_{3k}}L_{F_{3k}}=\frac{1}{5}\left(L_{F_{3k+3}}-L_{F_{3k-3}}\right).$$

Thus,

$$\sum_{k=1}^{n} F_{F_{3k}}F_{L_{3k}}L_{F_{3k}}=\frac{1}{5}\left(L_{F_{3n+3}}+L_{F_{3n}}-L_{F_3}-L_{F_0}\right)=\frac{1}{5}\left(L_{F_{3n+3}}+L_{F_{3n}}\right)-1.$$

Because $F_k\equiv 1 \pmod 2$ whenever $k\not\equiv 0 \pmod 3$,

$$\frac{1}{5}\left(L_{F_{3n+3}}+L_{F_{3n}}\right)=\frac{1}{5}\left(L_{F_{3n+2}+F_{3n+1}}-(-1)^{F_{3n+1}}L_{F_{3n+2}-F_{3n+1}}\right)=F_{F_{3n+1}}F_{F_{3n+2}};$$

hence,

$$\sum_{k=1}^{n} F_{F_{3k}}F_{L_{3k}}L_{F_{3k}}=F_{F_{3n+1}}F_{F_{3n+2}}-1.$$

Also solved by Thomas Achammer, Michel Bataille, High School Summer Research Group at the Citadel (Ethan Curb, Peyton Matheson, Aiden Milligan, Cameron Moening, Virginia Rhett Smith, and Ell Torek), Dimtry Fleischman, Robert Frontczak, G. C. Greubel, Wei-Kai Lai, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.