ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2021. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1286 Proposed by Michel Bataille, Rouen, France.

Let n be a positive integer. Prove that

$$\frac{\displaystyle\sum_{j=0}^{n} \binom{2n+1}{2j+1} \frac{1}{5^j}}{\displaystyle\sum_{j=0}^{n-1} \binom{2n}{2j+1} \frac{1}{5^j}} = \frac{2L_{2n+1}}{5F_{2n}}.$$

B-1287 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Define the sequence $\{G_n\}$ by $G_{n+2} = G_{n+1} + G_n$ for $n \geq 1$, with arbitrary G_1 and G_2. For integers $n \geq 1$ and $r \geq 2$, find a closed form expression for the sum

$$\sum_{k=1}^{n} \frac{G_{rk}}{F_{r-1}^k}.$$

B-1288 Proposed by Peter Ferraro, Roselle Park, NJ.

Prove that, for $n \geq 4$, if $F_{n+1}F_n$ is not a prefect square, then

$$\left\lfloor \sqrt{F_{n+1}F_n} \right\rfloor = \left\lfloor \sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}} \right\rfloor.$$

B-1289 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let x, y, and z be positive integers that satisfy the equation $F_{3n+2}\,x + F_{3n}\,y = F_{3n+1}\,z$. For every positive integer n, prove that $\sum_{k=1}^{3n} F_k^2$ and $2\sum_{k=1}^{3n+1} F_k^2$ are divisors of the product $(x+y)(y+z)(z-x)$.

B-1290 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Show that

$$\sum_{k=1}^{n} \left(5F_{2^k}^4 + 3F_{2^k}^2\right) = \left(F_{2^{n+1}} - 1\right)\left(F_{2^{n+1}} + 1\right).$$

SOLUTIONS

The Fourth Oldie from the Vault

B-416 Proposed by Gene Jakubowski and V. E. Hoggatt Jr., San Jose State University, San Jose, CA.
(Vol. 17.4, December 1979)

Let F_n be defined for all integers (positive, negative, and zero) by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, and hence

$$F_n = F_{n+2} - F_{n+1}.$$

Prove that every positive integer m has at least one representation of the form

$$m = \sum_{j=-N}^{N} \alpha_j F_j,$$

with each α_j in $\{0, 1\}$ and $\alpha_j = 0$ when j is an integral multiple of 3.

Solution by Bumkyu Cho, Durkbin Cho, Yung Duk Cho, Ho Park, and Joonsang Park, Dongguk University, Seoul, South Korea.

According to Zeckendorf's Theorem, m can be expressed as

$$m = F_{i_1} + F_{i_2} + \cdots + F_{i_r},$$

in which

$$2 \leq i_1 < i_2 < \cdots < i_r, \quad \text{where } i_{p+1} - i_p \geq 2 \text{ for } p = 1, 2, \ldots, r-1. \tag{1}$$

Starting with $I_0 = \{i_1, i_2, \ldots, i_r\}$, we want to transform it into another index set $I_1 = \{i'_1, i'_2, \ldots, i'_s\} \subset \mathbb{Z}$ such that

(i) $m = \sum_{i\in I_1} F_i$,

(ii) if $i \in I_1$ is odd, then $i < 0$, and

(iii) $6t + 3 \notin I_1$ for any integer t.

If $6t \pm 1 \in I_0$ for some integer $t \geq 0$, using $F_{-k} = (-1)^{k+1}F_k$, we can replace $6t \pm 1$ in I_0 with $-(6t \pm 1)$. When $6t + 3 \in I_0$ for some integer $t \geq 0$, we have

$$F_{6t+3} = -F_{6t+2} + F_{6t+4} = F_{-(6t+2)} + F_{6t+4},$$

Replace $6t + 3$ in I_0 with $-(6t + 2)$ and $6t + 4$, where $6t + 4 \notin I_0$ because of (1). Thus we obtain the desired index set I_1. Note that we may assume that if $2k \in I_1$ for some integer $k \neq 0$, then $-2k \notin I_1$ because $F_{2k} + F_{-2k} = 0$ implies that we can remove both $2k$ and $-2k$ from I_1.

Now, if $6t \notin I_1$ for any positive integer t, we are done. Otherwise, let n be the largest positive integer such that $6n \in I_1$. Let l, where $0 \leq l < 3n$, be the non-negative integer such that $6n-2, 6n-4, \ldots, 6n-2l \in I_1$, but $6n-2l-2 \notin I_1$. If $l = 0$, then $6n-1 \notin I_1$ by (ii), and $6n-2 \notin I_1$ by the choice of l. Thus, using $F_{6n} = F_{6n-1} + F_{6n-2}$, we can replace $6n$ in I_1 with $6n-1$ and $6n-2$. If $l > 0$, the sum of $6n, 6n-2, \ldots, 6n-2l$ becomes

$$\begin{aligned}\sum_{k=0}^{l} F_{6n-2l+2k} &= -F_{6n-2l} + F_{6n-2l} + \sum_{k=0}^{l} F_{6n-2l+2k}\\ &= -F_{6n-2l} + F_{6n-2l-2} + F_{6n-2l-1} + \sum_{k=0}^{l} F_{6n-2l+2k}\\ &= F_{-(6n-2l)} + F_{6n-2l-2} + F_{6n+1}.\end{aligned}$$

Here, $-(6n-2l) \notin I_1$ because $6n-2l \in I_1$, $6n-2l-2 \notin I_1$ by the choice of l, and $6n+1 \notin I_1$ because of (ii). Hence, we can replace $6n, 6n-2, \ldots, 6n-2l$ in I_1 with $-(6n-2l)$, $6n-2l-2$ and $6n+1$. It is possible that $-(6n-2l) = -6s$ for some positive integer s. In this case,

$$F_{-6s} = F_{-6s-1} + F_{-6s-2} = F_{6s+1} + F_{-(6s+2)},$$

where $6s+1 \notin I_1$ by (ii), and $-(6s+2) \notin I_1$ because $6s+2 = 6n-2l+2 \in I_1$.

By repeating the removal process above (note that if $6n-2l-2 = 6s$ for some positive integer s during any iteration, it becomes the largest positive multiple of 6 to be removed in the next iteration), we establish the desired expression for m.

Also solved by Raphael Schumacher (student), Doğa Can Sertbaş, Albert Stadler, David Terr, and the proposers.

A Nested Radical of Ones

B-1266 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 58.2, May 2020)

For any positive integer n, prove that

$$\frac{F_{2n}}{F_{2n-1}} \geq \sqrt{1+\sqrt{1+\sqrt{1+\cdots+\sqrt{1+\sqrt{1}}}}} \qquad (n \text{ square roots}).$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC.

For each positive integer n, we let

$$a_n = \sqrt{1+\sqrt{1+\sqrt{1+\cdots+\sqrt{1+\sqrt{1}}}}} \qquad (n \text{ square roots}).$$

Using the observation that $a_{n+1} = \sqrt{1+a_n}$, we use induction on n to prove that $F_{2n}/F_{2n-1} \geq a_n$. For $n = 1$, we have $F_2/F_1 = 1 = a_1$. Next, assume $F_{2n}/F_{2n-1} \geq a_n$ for some positive integer n. We must show that $F_{2n+2}/F_{2n+1} \geq a_{n+1}$, or equivalently that

$$F_{2n+2}^2 \geq F_{2n+1}^2(1+a_n).$$

Using the induction hypothesis, we establish this result by proving that

$$F_{2n+2}^2 \geq F_{2n+1}^2\left(1+\frac{F_{2n}}{F_{2n-1}}\right).$$

This is equivalent to establishing $F_{2n+2}^2 F_{2n-1} \geq F_{2n+1}^2(F_{2n-1}+F_{2n}) = F_{2n+1}^3$. Applying the identities $F_{2n+2}F_{2n-1} = F_{2n+1}F_{2n}+1$ and $F_{2n+2}F_{2n} = F_{2n+1}^2 - 1$, we obtain

$$F_{2n+2}^2 F_{2n-1} = F_{2n+2}(F_{2n+1}F_{2n}+1) = F_{2n+1}(F_{2n+1}^2-1) + F_{2n+2} = F_{2n+1}^3 + F_{2n},$$

which completes the proof.

We note that $\alpha \geq F_{2n}/F_{2n-1} \geq a_n$ for each positive integer n, with

$$\lim_{n\to\infty} \frac{F_{2n}}{F_{2n-1}} = \lim_{n\to\infty} a_n = \alpha.$$

Also solved by Michel Bataille, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Kapil Kumar Gurjar, Dongsheng Li (student), Luke Paluso (student), Ángel Plaza, Raphael Schumacher (student), J. N. Senadherra, Albert Stadler, David Terr, Dan Weiner, and the proposer.

The Zeta Riemann Function and the Cotangent Function

B-1267 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.
(Vol. 58.2, May 2020)

Prove that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)F_{2n}}{5^n} = -1 + \sum_{n=1}^{\infty} \frac{\zeta(2n)L_{2n}}{5^n} = \frac{\pi}{2\sqrt{5}} \tan\left(\frac{\pi}{2\sqrt{5}}\right),$$

where $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$ $(s > 1)$ is the Riemann zeta function.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

First,

$$\sum_{n=1}^{\infty} \zeta(2n)x^{2n} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^{2n}}{k^{2n}} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \left(\frac{x^2}{k^2}\right)^n = \sum_{k=1}^{\infty} \frac{x^2}{k^2-x^2} = \frac{1}{2}\left(1-\pi x\cot \pi x\right).$$

Next,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta(2n)F_{2n}}{5^n} &= \frac{1}{\sqrt{5}}\left[\sum_{n=1}^{\infty}\zeta(2n)\left(\frac{\alpha}{\sqrt{5}}\right)^{2n}-\sum_{n=1}^{\infty}\zeta(2n)\left(\frac{\beta}{\sqrt{5}}\right)^{2n}\right]\\ &= \frac{\pi}{2\sqrt{5}}\left(\frac{\beta}{\sqrt{5}}\cot\frac{\pi\beta}{\sqrt{5}}-\frac{\alpha}{\sqrt{5}}\cot\frac{\pi\alpha}{\sqrt{5}}\right).\end{aligned}$$

With $\alpha-\beta=\sqrt{5}$, it follows that $\frac{\beta}{\sqrt{5}}=\frac{\alpha}{\sqrt{5}}-1$, and

$$\frac{\beta}{\sqrt{5}}\cot\frac{\pi\beta}{\sqrt{5}}=\left(\frac{\alpha}{\sqrt{5}}-1\right)\cot\left(\frac{\pi\alpha}{\sqrt{5}}-\pi\right)=\left(\frac{\alpha}{\sqrt{5}}-1\right)\cot\frac{\pi\alpha}{\sqrt{5}}.$$

Hence,

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)F_{2n}}{5^n}=-\frac{\pi}{2\sqrt{5}}\cot\frac{\pi\alpha}{\sqrt{5}}=-\frac{\pi}{2\sqrt{5}}\cot\left(\frac{\pi}{2\sqrt{5}}+\frac{\pi}{2}\right)=\frac{\pi}{2\sqrt{5}}\tan\left(\frac{\pi}{2\sqrt{5}}\right).$$

For the summation involving the Lucas numbers:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta(2n)L_{2n}}{5^n} &= \sum_{n=1}^{\infty}\zeta(2n)\left(\frac{\alpha}{\sqrt{5}}\right)^{2n}+\sum_{n=1}^{\infty}\zeta(2n)\left(\frac{\beta}{\sqrt{5}}\right)^{2n}\\ &= 1-\frac{\pi}{2\sqrt{5}}\left(\alpha\cot\frac{\pi\alpha}{\sqrt{5}}+\beta\cot\frac{\pi\beta}{\sqrt{5}}\right).\end{aligned}$$

Since

$$\beta\cot\frac{\pi\beta}{\sqrt{5}}=\left(\alpha-\sqrt{5}\right)\cot\frac{\pi\alpha}{\sqrt{5}}=(1-\alpha)\cot\frac{\pi\alpha}{\sqrt{5}},$$

we deduce that

$$-1+\sum_{n=1}^{\infty}\frac{\zeta(2n)L_{2n}}{5^n}=-\frac{\pi}{2\sqrt{5}}\cot\frac{\pi\alpha}{\sqrt{5}}=\frac{\pi}{2\sqrt{5}}\tan\left(\frac{\pi}{2\sqrt{5}}\right).$$

Editor's Note: Paluso obtained a generalized result

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)G_{2n}}{5^n}=\frac{1}{2\sqrt{5}}\left[a\pi\tan\left(\frac{\pi}{2\sqrt{5}}\right)-(a-b)\sqrt{5}\right]$$

for the sequence $\{G_n\}_{n\geq 1}$ defined by $G_1=a$, $G_2=b$, and $G_n=G_{n-1}+G_{n-2}$.

Also solved by Michel Bataille, Dmitry Fleischman, Kapil Kumar Gurjar, Luke Paluso (student), Ángel Plaza, Raphael Schumacher (student), J. N. Senadherra, Jason L. Smith, Albert Stadler, David Terr, and the proposer.

<u>Prove It in Any Way You Like</u>

<u>B-1268</u> Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain. (Vol. 58.2, May 2020)

Prove that, for $n\geq 1$,

(i) $L_{2n-1}=L_{2n-3}+2L_{2n-5}+\cdots+(n-1)L_1+2n-1$

(ii) $L_{2n}=L_{2n-2}+2L_{2n-4}+\cdots+(n-1)L_2+n+2$

Solution 1 by Michel Bataille, Rouen, France.

Note that (i) and (ii) hold if $n = 1$ so we may assume that $n \geq 2$ in what follows. If m is a positive integer, then for $k = 1, 2, \ldots, m$, we have $L_{2k} - L_{2k-2} = L_{2k-1}$, hence

$$L_{2m} - 2 = L_{2m} - L_0 = \sum_{k=1}^{m} (L_{2k} - L_{2k-2}) = L_1 + L_3 + \cdots + L_{2m-1}. \tag{2}$$

Similarly, if $m \geq 2$, then from $L_{2k-1} - L_{2k-3} = L_{2k-2}$ $(k = 2, \ldots, m)$, we obtain

$$L_{2m-1} - 1 = L_{2m-1} - L_1 = \sum_{k=2}^{m} (L_{2k-1} - L_{2k-3}) = L_2 + L_4 + \cdots + L_{2m-2}. \tag{3}$$

From (2), we have

$$\begin{aligned} \sum_{m=1}^{n-1} (L_{2m} - 2) &= L_1 + (L_1 + L_3) + \cdots + (L_1 + L_3 + \cdots + L_{2n-5}) \\ &\quad + (L_1 + L_3 + \cdots + L_{2n-3}). \end{aligned}$$

Then, using (3),

$$L_{2n-1} - 1 - 2(n-1) = (n-1)L_1 + (n-2)L_3 + \cdots + 2L_{2n-5} + L_{2n-3},$$

and (i) immediately follows. Analogously, it follows from (3) that

$$\begin{aligned} \sum_{m=2}^{n} (L_{2m-1} - 1) &= L_2 + (L_2 + L_4) + \cdots + (L_2 + L_4 + \cdots + L_{2n-4}) \\ &\quad + (L_2 + L_4 + \cdots + L_{2n-2}). \end{aligned}$$

Then, using (2),

$$L_{2n} - 2 - L_1 - (n-1) = (n-1)L_2 + (n-2)L_4 + \cdots + 2L_{2n-4} + L_{2n-2},$$

and (ii) follows.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA.

We first consider (ii), and rewrite it as

$$L_{2n} = \sum_{k=0}^{n} kL_{2(n-k)} - n + 2.$$

Using the generating functions

$$\sum_{n=0}^{\infty} L_{2n}x^n = \frac{2-3x}{1-3x+x^2}, \qquad \sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}, \qquad \sum_{n=1}^{\infty} x^n = \frac{1}{1-x},$$

and applying convolution, we find

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} kL_{2(n-k)} - n + 2 \right) x^n = \frac{2-3x}{1-3x+x^2} \cdot \frac{x}{(1-x)^2} - \frac{x}{(1-x)^2} + \frac{2}{1-x}.$$

After simplification, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} kL_{2(n-k)} - n + 2 \right) x^n = \frac{2-3x}{1-3x+x^2} = \sum_{n=0}^{\infty} L_{2n}x^n.$$

This proves (ii). Now, subtract

$$L_{2n-2} = L_{2n-4} + 2L_{2n-6} + \cdots + (n-1)L_0 - (n-1) + 2$$

from

$$L_{2n} = L_{2n-2} + 2L_{2n-4} + \cdots + (n-1)L_2 + nL_0 - n + 2$$

to obtain (i).

Solution 3 by the proposer.

We will use a combinatorial argument similar to that given in [1]. It is well-known that L_{2n-1} on the left side of (i) counts the number of ways to tile a *labeled* circular board of length $2n-1$ with squares and dominoes. For the right side, the non-homogeneous term $2n-1$ counts the tilings consisting of a single square that can be located in any of the $2n-1$ cells, with $n-1$ dominoes. If a tiling has at least two squares, the location of the *second* square depends on how the first and last cells are tiled.

- If the tiling is *out of phase*, meaning that cell $2n-1$ and cell 1 are covered by a domino, then the second square appears on cell $2j+1$, where $1 \le j \le n-2$. Removing the $2j$ tiles covering cells 2 through $2j+1$ yields a tiling of a circular $(2n-1-2j)$-board of the same type.
- If the tiling is *in-phase*, meaning that cell $2n-1$ and cell 1 are not covered by a domino, then the second square appears on cell $2j$, where $1 \le j \le n-1$. Removing the $2j$ tiles covering cells 1 through $2j$ yields a tiling of a circular $(2n-1-2j)$-board of the same type. Note that when $j = n-1$, only one cell is left after $2(n-1)$ cells are removed, and there is only $L_1 = 1$ way to tile a single cell.

In both cases, there are j ways to place the first square. Because there are $L_{2n-1-2j}$ ways to tile a circular $(2n-1-2j)$-board, we deduce that

$$L_{2n-1} = 2n - 1 + \left(\sum_{j=1}^{n-2} jL_{2n-1-2j}\right) + (n-1)L_1 = 2n - 1 + \sum_{j=1}^{n-1} jL_{2n-1-2j},$$

which proves (i).

The proof of (ii) follows from (i) by taking the difference of the identities for L_{2n+1} and L_{2n-1} and using the recurrence $L_{k+1} - L_{k-1} = L_k$.

Editor's Notes: Using induction, Ohtsuka showed that, for any integers r and n with $n \ge 1$,

$$\sum_{k=1}^{n-1}(n-k)L_{2k+r} = L_{2n+r} - nL_{r+1} - L_r. \tag{4}$$

Substituting in $r = -1$ and $r = 0$ yield the identities (i) and (ii). Ohtsuka also reported that (4) can be further generalized: for any integers r, n and p with $n \ge 1$ and $p \ne 0$,

$$\sum_{k=1}^{n-1}(n-k)G_{2k+r} = \frac{1}{p^2}\left(G_{2n+r} - npG_{r+1} - G_r\right),$$

where the sequence $\{G_n\}$ satisfies $G_{n+2} = pG_{n+1} + G_n$ for all integers n.

Reference

[1] A. T. Benjamin, J. Crouch, and J. A. Sellers, *Unified tiling proofs of a family of Fibonacci identities*, The Fibonacci Quarterly, **57.1** (2019), 29–31.

Also solved by Steve Edwards, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Kapil Kumar Gurjar, Carl Libis, Hideyuki Ohtsuka, Luke Paluso (student), Hemlatha Rajpurohit, Raphael Schumacher (student), J. N. Senadherra, Jason L. Smith, Allbert Stadler, Daniel Văcaru, and the proposer.

An Inequality in Two Variables

B-1269 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 58.2, May 2020)

For all integers n and real numbers $x \leq y$, prove that

$$L_{n-1}(xF_n + yF_{n+2}) \leq xF_{n-2}F_{n+2} + 4yF_n^2.$$

Solution by Raphael Schumacher (student), ETH Zurich, Switzerland.

The given inequality is equivalent to

$$x(L_{n-1}F_n - F_{n-2}F_{n+2}) + y(L_{n-1}F_{n+2} - 4F_n^2) \leq 0.$$

Using the two identities $L_{n-1} = 2F_n - F_{n-1}$ and $F_{n+2} = 2F_n + F_{n-1}$, we obtain

$$L_{n-1}F_n - F_{n-2}F_{n+2} = (2F_n - F_{n-1})F_n - (F_n - F_{n-1})(2F_n + F_{n-1}) = F_{n-1}^2,$$
$$L_{n-1}F_{n+2} - 4F_n^2 = (2F_n - F_{n-1})(2F_n + F_{n-1}) - 4F_n^2 = -F_{n-1}^2.$$

Therefore,

$$x(L_{n-1}F_n - F_{n-2}F_{n+2}) + y(L_{n-1}F_{n+2} - 4F_n^2) = (x-y)F_{n-1}^2 \leq 0.$$

Also solved by Michel Bataille, Brian D. Beasley, Brian Bradie, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Luke Paluso (student), Ángel Plaza, J. N. Senadherra, Albert Stadler, Dan Weiner, and the proposer.

Four Telescopic Infinite Products

B-1270 Pridon Davlianidze, Tbilisi, Republic of Georgia.
(Vol. 58.2, May 2020)

Evaluate the following infinite products:

(A) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{5}{L_{2n-1}^2}\right)$ (B) $\displaystyle\prod_{n=2}^{\infty}\left(1 + \frac{5}{L_{2n}^2}\right)$

(C) $\displaystyle\prod_{n=2}^{\infty}\left(1 + \frac{5}{L_{2n-1}^2}\right)\left(1 - \frac{5}{L_{2n}^2}\right)$ (D) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{25}{L_n^4}\right)$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

We will use the product formula $L_mL_n = L_{m+n} + (-1)^n L_{m-n}$ [1] in the following proofs. For (A), observe that

$$L_{2n-1}^2 - L_{2n+1}L_{2n-3} = (L_{4n-2} - L_0) - (L_{4n-2} - L_4) = 5.$$

Then,

$$\prod_{n=2}^{\infty}\left(1-\frac{5}{L_{2n-1}^2}\right)=\lim_{m\to\infty}\prod_{n=2}^{m}\frac{L_{2n-1}^2-5}{L_{2n-1}^2}=\lim_{m\to\infty}\prod_{n=2}^{m}\frac{L_{2n-3}L_{2n+1}}{L_{2n-1}^2}.$$

This product telescopes, leaving

$$\prod_{n=2}^{\infty}\left(1-\frac{5}{L_{2n-1}^2}\right)=\lim_{m\to\infty}\frac{L_1L_{2m+1}}{L_3L_{2m-1}}=\frac{\alpha^2}{4}.$$

For (B), we see that

$$L_{2n}^2-L_{2n+2}L_{2n-2}=(L_{4n}+L_0)-(L_{4n}+L_4)=-5,$$

so that

$$\prod_{n=2}^{\infty}\left(1+\frac{5}{L_{2n}^2}\right)=\lim_{m\to\infty}\prod_{n=2}^{m}\frac{L_{2n-2}L_{2n+2}}{L_{2n}^2}=\lim_{m\to\infty}\frac{L_2L_{2m+2}}{L_4L_{2m}}=\frac{3\alpha^2}{7}.$$

(C) Again, we use the product formula to obtain

$$L_{2n-1}^2-L_{2n}F_{2n-2}=(L_{4n-2}-L_0)-(L_{4n-2}+L_2)=-5,$$
$$L_{2n}^2-L_{2n+1}L_{2n-1}=(L_{4n}+L_0)-(L_{4n}-L_2)=5.$$

Our product becomes

$$\lim_{m\to\infty}\prod_{n=2}^{m}\left(\frac{L_{2n-2}L_{2n}}{L_{2n-1}^2}\right)\left(\frac{L_{2n-1}L_{2n+1}}{L_{2n}^2}\right)=\lim_{m\to\infty}\prod_{n=2}^{m}\frac{L_{2n-2}L_{2n+1}}{L_{2n-1}L_{2n}}=\lim_{m\to\infty}\frac{L_2L_{2m+1}}{L_3L_{2m}}=\frac{3\alpha}{4}.$$

For (D), factor the general term inside the product and then separate into even and odd-indexed terms. The infinite product becomes

$$\left(1-\frac{5}{L_2^2}\right)\left(1+\frac{5}{L_2^2}\right)\cdot\prod_{k=2}^{\infty}\left(1-\frac{5}{L_{2k-1}^2}\right)\cdot\prod_{k=2}^{\infty}\left(1+\frac{5}{L_{2k}^2}\right)\cdot\prod_{k=2}^{\infty}\left(1+\frac{5}{L_{2k-1}^2}\right)\left(1-\frac{5}{L_{2k}^2}\right).$$

This enables us to use the three results above. We determine that

$$\prod_{n=2}^{\infty}\left(1-\frac{25}{L_n^4}\right)=\frac{4}{9}\cdot\frac{14}{9}\cdot\frac{\alpha^2}{4}\cdot\frac{3\alpha^2}{7}\cdot\frac{3\alpha}{4}=\frac{\alpha^5}{18}.$$

Editor's Note: The identities (A), (B), and (C) are Lucas analogs of Problem B-1264, and (D) is the Lucas analog of Problem B-1620.

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

Also solved by Michel Bataille, Brian D. Beasley, Brian Bradie, Charles K. Cook, Kenny B. Davenport, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Thomas Koshy, Carl Libis, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (student), J. N. Senadherra, Albert Stadler, David Terr, and the proposer.

Belated Acknowledgment: Michel Bataille's name was inadvertently omitted from the list of solvers of Problems B-1261–B-1265. The editor would like to express his sincere apologies for his oversight.

Correction: At the very end of the solution of Problem B-1265, replace $L_{2^{n+3}}$ with $L_{2^{n+2}}$.