ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2022. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1306 **Proposed by Diego Rattaggi, Realgymnasium Rämibühl, Zürich, Switzerland.**

Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} F_{6n}}{(F_{2n}^2+1)^2} = 1.$$

B-1307 **Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.**

Show that

$$\sum_{n=0}^{\infty} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \frac{F_n}{2^n} = \frac{32}{5} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \frac{L_n}{2^n} = 16.$$

B-1308 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{L_{2F_n} L_{2F_{n+1}} L_{2F_{n+2}} L_{2F_{n+3}}} \left(\frac{1}{L_{2F_n}} - \frac{1}{L_{2F_{n+3}}} \right).$$

B-1309 Proposed by Kenny B. Davenport, Dallas, PA.

Prove that, for any integer $n \geq 1$,

$$\sum_{k=1}^{n} F_k F_{k+1}^2 = \frac{3F_n^3 + 3F_{n+1}^3 + F_{n+2}^3 - F_{3n+2} - 3}{6}.$$

B-1310 Proposed by Steve Edwards, Roswell, GA.

For any positive integer n, find a closed form expression for the sum

$$\sum_{k=1}^{n} \left\lfloor \frac{F_k}{\alpha F_k - F_{k-1}} \right\rfloor.$$

SOLUTIONS

Summing the Odd Terms in a Binomial Expansion

B-1286 Proposed by Michel Bataille, Rouen, France.
(Vol. 59.2, May 2021)

Let n be a positive integer. Prove that

$$\frac{\sum_{j=0}^{n} \binom{2n+1}{2j+1} \frac{1}{5^j}}{\sum_{j=0}^{n-1} \binom{2n}{2j+1} \frac{1}{5^j}} = \frac{2L_{2n+1}}{5F_{2n}}.$$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

We can use the binomial theorem to obtain

$$\sum_{j=0}^{n} \binom{2n+1}{2j+1} x^{2j+1} = \frac{1}{2} \left[(1+x)^{2n+1} - (1-x)^{2n+1} \right].$$

If we put $x = \frac{1}{\sqrt{5}}$, we find

$$\begin{aligned} \sum_{j=0}^{n} \binom{2n+1}{2j+1} \frac{1}{5^j} &= \frac{\sqrt{5}}{2} \left[\left(1 + \frac{1}{\sqrt{5}}\right)^{2n+1} - \left(1 - \frac{1}{\sqrt{5}}\right)^{2n+1} \right] \\ &= \frac{\sqrt{5}}{2} \left[\left(\frac{2\alpha}{\sqrt{5}}\right)^{2n+1} - \left(-\frac{2\beta}{\sqrt{5}}\right)^{2n+1} \right] \\ &= \left(\frac{4}{5}\right)^n L_{2n+1}. \end{aligned}$$

Similarly, we have

$$\sum_{j=0}^{n-1} \binom{2n}{2j+1} x^{2j+1} = \frac{1}{2}\left[\,(1+x)^{2n} - (1-x)^{2n}\,\right].$$

Again, put $x = \frac{1}{\sqrt{5}}$ to find

$$\begin{aligned}\sum_{j=0}^{n-1} \binom{2n}{2j+1} \frac{1}{5^j} &= \frac{\sqrt{5}}{2}\left[\left(1+\frac{1}{\sqrt{5}}\right)^{2n} - \left(1-\frac{1}{\sqrt{5}}\right)^{2n}\right] \\ &= \frac{\sqrt{5}}{2}\left[\left(\frac{2\alpha}{\sqrt{5}}\right)^{2n} - \left(-\frac{2\beta}{\sqrt{5}}\right)^{2n}\right] \\ &= \frac{5}{2}\left(\frac{4}{5}\right)^{n} F_{2n}.\end{aligned}$$

Finally, the desired result follows:

$$\frac{\sum_{j=0}^{n} \binom{2n+1}{2j+1}\frac{1}{5^j}}{\sum_{j=0}^{n-1} \binom{2n}{2j+1}\frac{1}{5^j}} = \frac{\left(\frac{4}{5}\right)^n L_{2n+1}}{\frac{5}{2}\left(\frac{4}{5}\right)^n F_{2n}} = \frac{2L_{2n+1}}{5F_{2n}}.$$

Also solved by Thomas Achammer, Brian Bradie, Nandan Sai Dasireddy, Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Haydn Gwyn (undergraduate), Kapil Kumar, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, Seán M. Stewart, David Terr, Andrés Ventas, and the proposer.

A Summation of Generalized Fibonacci Numbers

B-1287 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 59.2, May 2021)

Define the sequence $\{G_n\}$ by $G_{n+2} = G_{n+1} + G_n$ for $n \geq 1$, with arbitrary G_1 and G_2. For integers $n \geq 1$ and $r \geq 2$, find a closed form expression for the sum

$$\sum_{k=1}^{n} \frac{G_{rk}}{F_{r-1}^{k}}.$$

Solution 1 by Raphael Schumacher (graduate student), ETH Zurich, Switzerland.

For all integers $n \geq 1$ and $a \geq 0$, we have the identity

$$G_{n+a} = F_{a+1}G_n + F_a G_{n-1},$$

which can be proved by a double induction based on

$$\begin{aligned}G_{n+(a+1)} &= G_{n+a} + G_{n+(a-1)} \\ &= (F_{a+1}G_n + F_a G_{n-1}) + (F_a G_n + F_{a-1}G_{n-1}) \\ &= F_{a+2}G_n + F_{a+1}G_{n-1}.\end{aligned}$$

From the above identity for G_{n+a}, it follows that

$$G_{rk+r-1} = F_r G_{rk} + F_{r-1} G_{rk-1},$$

which implies, after dividing both sides by $F_rF_{r-1}^k$, the identity

$$\frac{G_{rk}}{F_{r-1}^k} = \frac{G_{rk+r-1}}{F_rF_{r-1}^k} - \frac{G_{rk-1}}{F_rF_{r-1}^{k-1}}$$

for all integers $k \geq 1$. By telescoping it follows from this identity that

$$\sum_{k=1}^{n} \frac{G_{rk}}{F_{r-1}^k} = \sum_{k=1}^{n} \left(\frac{G_{rk+r-1}}{F_rF_{r-1}^k} - \frac{G_{rk-1}}{F_rF_{r-1}^{k-1}} \right) = \frac{G_{rn+r-1}}{F_rF_{r-1}^n} - \frac{G_{r-1}}{F_r}.$$

Solution 2 by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

We shall prove a generalization. Let m, r, and s be integers with m, s arbitrary, $r \neq 0$, such that $m + r \geq 1$. Then, we have the following identity:

$$\sum_{k=1}^{n} \frac{F_m^k G_{rk+s}}{F_{m+r}^k} = (-1)^{m+1} \left(\frac{F_m^{n+1} G_{rn+r+s+m}}{F_rF_{m+r}^n} - \frac{F_m G_{r+s+m}}{F_r} \right).$$

In particular, if $m = -1$, then (since $F_{-1} = 1$) we have for all $r \geq 2$,

$$\sum_{k=1}^{n} \frac{G_{rk+s}}{F_{r-1}^k} = \frac{G_{rn+r+s-1}}{F_rF_{r-1}^n} - \frac{G_{r+s-1}}{F_r}.$$

To prove the generalized result, we start with the Binet formula

$$G_n = A\alpha^n + B\,\beta^n, \qquad \text{where } A = \frac{G_1 - G_0\beta}{\alpha - \beta} \text{ and } B = \frac{G_0\alpha - G_1}{\alpha - \beta}.$$

We deduce that

$$\begin{aligned} \sum_{k=1}^{n} \frac{F_m^k G_{rk+s}}{F_{m+r}^k} &= A\alpha^s \sum_{k=1}^{n} \left(\frac{\alpha^r F_m}{F_{m+r}} \right)^k + B\beta^s \sum_{k=1}^{n} \left(\frac{\beta^r F_m}{F_{m+r}} \right)^k \\ &= \frac{A\alpha^{r+s} F_m \left[\left(\frac{\alpha^r F_m}{F_{m+r}} \right)^n - 1 \right]}{F_{m+r} \left(\frac{\alpha^r F_m}{F_{m+r}} - 1 \right)} + \frac{B\beta^{r+s} F_m \left[\left(\frac{\beta^r F_m}{F_{m+r}} \right)^n - 1 \right]}{F_{m+r} \left(\frac{\beta^r F_m}{F_{m+r}} - 1 \right)}. \end{aligned}$$

It follows easily from the Binet form for Fibonacci numbers that $\alpha^r F_m + \beta^m F_r = F_{m+r}$. Hence,

$$F_{m+r} \left(\frac{\alpha^r F_m}{F_{m+r}} - 1 \right) = -\beta^m F_r = (-1)^{m+1} \frac{F_r}{\alpha^m},$$

and, due to symmetry,

$$F_{m+r} \left(\frac{\beta^r F_m}{F_{m+r}} - 1 \right) = -\alpha^m F_r = (-1)^{m+1} \frac{F_r}{\beta^m}.$$

Then,

$$\begin{aligned} \sum_{k=1}^{n} \frac{F_m^k G_{rk+s}}{F_{m+r}^k} &= (-1)^{m+1} \left[\frac{A\alpha^{r+s+m} F_m}{F_r} \left(\frac{\alpha^{rn} F_m^n}{F_{m+r}^n} - 1 \right) + \frac{B\beta^{r+s+m} F_m}{F_r} \left(\frac{\beta^{rn} F_m^n}{F_{m+r}^n} - 1 \right) \right] \\ &= (-1)^{m+1} \left(\frac{F_m^{n+1} G_{rn+r+s+m}}{F_rF_{m+r}^n} - \frac{F_m G_{r+s+m}}{F_r} \right). \end{aligned}$$

In closing, we note that if $m = r$, then we have for all $r \geq 0$,

$$\sum_{k=1}^{n} \frac{G_{rk+s}}{L_r^k} = (-1)^{r+1} \left(\frac{G_{rn+2r+s}}{L_r^n} - G_{2r+s} \right),$$

which can be considered as the Lucas counterpart of Problem B-1287.

Also solved by Michel Bataille, Brian Bradie, Steve Edwards, Dmitry Fleischman, G. C. Greubel, Haydn Gwyn (undergraduate), Ángel Plaza, Albert Stadler, Andrés Ventas, and the proposer.

An Identity in Floor Functions

B-1288 Proposed by Peter Ferraro, Roselle Park, NJ. (Vol. 59.2, May 2021)

Prove that, for $n \geq 4$, if $F_{n+1}F_n$ is not a prefect square, then

$$\left\lfloor \sqrt{F_{n+1}F_n} \right\rfloor = \left\lfloor \sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}} \right\rfloor.$$

Composite solution by the proposer and the Problems Section Editor.

We first use the product formulas $5F_sF_t = L_{s+t} - (-1)^t L_{s-t}$ and $L_sL_t = L_{s+t} + (-1)^t L_{s-t}$, and the identities $F_{t-1} + F_{t+1} = L_t$ and $L_{t-1} + L_{t+1} = 5F_t$ to establish two preliminary results.

Lemma 1. For all integers n,

$$L_{n-1}L_{n-2}F_{n-3}F_{n-4} = F_{2n-5}^2 - (-1)^n F_{2n-5} - 2.$$

Proof. The product formulas assert that $L_{n-1}L_{n-2} = L_{2n-3} + (-1)^n$ and $5F_{n-3}F_{n-4} = L_{2n-7} - (-1)^n$. Hence,

$$\begin{aligned} 5L_{n-1}L_{n-2}F_{n-3}F_{n-4} &= L_{2n-3}L_{2n-7} - (-1)^n(L_{2n-3} - L_{2n-7}) - 1 \\ &= L_{4n-10} - (-1)^n(L_{2n-4} + L_{2n-6}) - 8 \\ &= 5F_{2n-5}^2 - 5(-1)^n F_{2n-5} - 10, \end{aligned}$$

from which the desired result follows. □

Lemma 2. For all integers n,

$$F_{n+1}F_n - F_{n-3}F_{n-4} - L_{n-1}L_{n-2} = 2F_{2n-5} - (-1)^n.$$

Proof. Using the product formulas, we have

$$\begin{aligned} 5(F_{n+1}F_n - F_{n-3}F_{n-4}) &= L_{2n+1} - L_{2n-7} \\ &= 3(L_{2n-2} + L_{2n-4}) \\ &= 3 \cdot 5F_{2n-3}. \end{aligned}$$

Thus, $F_{n+1}F_n - F_{n-3}F_{n-4} = 3F_{2n-3}$. Together with $L_{n-1}L_{n-2} = L_{2n-3} + (-1)^n$, we obtain

$$\begin{aligned} F_{n+1}F_n - F_{n-3}F_{n-4} - L_{n-1}L_{n-2} &= 3F_{2n-3} - F_{2n-2} - F_{2n-4} - (-1)^n \\ &= 2F_{2n-5} - (-1)^n. \end{aligned}$$

This completes the proof of the lemma. □

These two results lead to the following lemmas.

Lemma 3. For all integers $n \geq 3$,
$$\sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}} < \sqrt{F_{n+1}F_n + 1}.$$

Proof. It suffices to prove that
$$L_{n-1}L_{n-2} + F_{n-3}F_{n-4} + 2\sqrt{L_{n-1}L_{n-2}F_{n-3}F_{n-4}} < F_{n+1}F_n + 1,$$
or
$$(F_{n+1}F_n - F_{n-3}F_{n-4} - L_{n-1}L_{n-2} + 1)^2 > 4L_{n-1}L_{n-2}F_{n-3}F_{n-4}.$$
Because of Lemmas 1 and 2, the proof of Lemma 3 will be completed if we can show that
$$\big[2F_{2n-5} + 1 - (-1)^n\big]^2 > 4\big[F_{2n-5}^2 - (-1)^n F_{2n-5} - 2\big],$$
which simplifies to
$$4F_{2n-5} + [1 - (-1)^n]^2 > -8.$$
Because this inequality is obviously valid, the proof is complete. □

Lemma 4. For all integers $n \geq 5$,
$$\sqrt{F_{n+1}F_n - 1} < \sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}}.$$

Proof. It suffices to prove that
$$F_{n+1}F_n - 1 < L_{n-1}L_{n-2} + F_{n-3}F_{n-4} + 2\sqrt{L_{n-1}L_{n-2}F_{n-3}F_{n-4}},$$
or
$$(F_{n+1}F_n - F_{n-3}F_{n-4} - L_{n-1}L_{n-2} - 1)^2 < 4L_{n-1}L_{n-2}F_{n-3}F_{n-4}.$$
Because of Lemmas 1 and 2, the proof of Lemma 4 will be completed if we can show that
$$\big[2F_{2n-5} - 1 - (-1)^n\big]^2 < 4\big[F_{2n-5}^2 - (-1)^n F_{2n-5} - 2\big],$$
which simplifies to
$$4F_{2n-5} > 8 + [1 + (-1)^n]^2.$$
Because this inequality is valid when $n \geq 5$, the proof is complete. □

Proof of the Proposed Problem. It is easy to verify the identity when $n = 4$, so we may assume $n \geq 5$. Let
$$\left\lfloor \sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}} \right\rfloor = m,$$
and suppose $\left\lfloor \sqrt{F_{n+1}F_n} \right\rfloor \neq m$. Because $F_{n+1}F_n$ is not a perfect square, we would have either $\sqrt{F_{n+1}F_n} < m$, or $m + 1 < \sqrt{F_{n+1}F_n}$. If $\sqrt{F_{n+1}F_n} < m$, then by Lemma 3, we have
$$\sqrt{F_{n+1}F_n} < m \leq \sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}} < \sqrt{F_{n+1}F_n + 1}.$$
Thus,
$$F_{n+1}F_n < m^2 < F_{n+1}F_n + 1,$$
an impossibility. Now if $m + 1 < \sqrt{F_{n+1}F_n}$, then by Lemma 4,
$$\sqrt{F_{n+1}F_n - 1} < \sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}} < m + 1 < \sqrt{F_{n+1}F_n},$$
and so
$$F_{n+1}F_n - 1 < (m+1)^2 < F_{n+1}F_n,$$
again impossible. Hence,
$$\left\lfloor \sqrt{F_{n+1}F_n} \right\rfloor = m = \left\lfloor \sqrt{L_{n-1}L_{n-2}} + \sqrt{F_{n-3}F_{n-4}} \right\rfloor.$$

Editor's Notes: Schumacher remarked that the condition that $F_{n+1}F_n$ is not a perfect square can be removed, because $F_{n+1}F_n$ can never be a perfect square for $n \geq 2$. It is well-known (see the solution to Problem B-1289) that $\gcd(F_{n+1}, F_n) = 1$. Therefore, $F_{n+1}F_n$ can only be a perfect square if both F_{n+1} and F_n are perfect squares. This only occurs when $n = 0$ or $n = 1$, because the only Fibonacci squares are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, and $F_{12} = 144$ [1].

References

[1] J. H. E. Cohn, *On square Fibonacci numbers*, J. London Math. Soc., **39** (1964), 537–540.

Also solved by Haydn Gwyn (undergraduate), Raphael Schumacher, Albert Stadler, Andrés Ventas, and the proposer.

A Product of Three Consecutive Fibonacci Numbers

B-1289 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine. (Vol. 59.2, May 2021)

Let x, y, and z be positive integers that satisfy the equation $F_{3n+2}\,x + F_{3n}\,y = F_{3n+1}\,z$. For every positive integer n, prove that $\displaystyle\sum_{k=1}^{3n} F_k^2$ and $2\displaystyle\sum_{k=1}^{3n+1} F_k^2$ are divisors of the product $(x+y)(y+z)(z-x)$.

Solution by Michel Bataille, Rouen, France.

From the hypothesis, we have

$$\begin{aligned} x(F_{3n+1}+F_{3n}) + yF_{3n} &= zF_{3n+1}, \\ xF_{3n+2} + yF_{3n} &= z(F_{3n+2} - F_{3n}), \\ xF_{3n+2} + y(F_{3n+2} - F_{3n+1}) &= zF_{3n+1}. \end{aligned}$$

We obtain these relations:

$$\begin{aligned} (x+y)F_{3n} &= (z-x)F_{3n+1}, \\ (z-x)F_{3n+2} &= (y+z)F_{3n}, \\ (x+y)F_{3n+2} &= (y+z)F_{3n+1}. \end{aligned}$$

As the first relation shows, F_{3n} divides $(z-x)F_{3n+1}$. But, we know that $\gcd(F_{3n}, F_{3n+1}) = \gcd(3n, 3n+1) = 1$, hence F_{3n} divides $z - x$, and we can write $z - x = mF_{3n}$ for some integer m. Substitute this into the last set of equations yields $y + z = mF_{3n+2}$, and $x + y = mF_{3n+1}$. Therefore,

$$(x+y)(y+z)(z-x) = m^3 F_{3n}F_{3n+1}F_{3n+2}.$$

We deduce that $\sum_{k=1}^{3n} F_k^2 = F_{3n}F_{3n+1}$ divides $(x+y)(y+z)(z-x)$. In addition, because F_{3n} is even, we see that $2\sum_{k=1}^{3n+1} F_k^2 = 2F_{3n+1}F_{3n+2}$ also divides $(x+y)(y+z)(z-x)$.

Also solved by Thomas Achammer, Illia Antypenko (high school student), Brian D. Beasley, Brian Bradie, Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Gruebel, Haydn Gwyn (undergraduate), Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, Andrés Ventas, and the proposer.

The Magic of Powers of Two as Subscripts

B-1290 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany. (Vol. 59.2, May 2021)

Show that

$$\sum_{k=1}^{n}\left(5F_{2^k}^4+3F_{2^k}^2\right)=\left(F_{2^{n+1}}-1\right)\left(F_{2^{n+1}}+1\right).$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

By the fundamental identity, $L_n^2-5F_n^2=4(-1)^n$; for $k\ge 1$, it follows that

$$L_{2^k}^2-5F_{2^k}^2=4$$

and

$$\begin{aligned}5F_{2^k}^4+3F_{2^k}^2 &= F_{2^k}^2(5F_{2^k}^2+4)-F_{2^k}^2\\ &= F_{2^k}^2L_{2^k}^2-F_{2^k}^2\\ &= F_{2\cdot 2^k}^2-F_{2^k}^2\\ &= F_{2^{k+1}}^2-F_{2^k}^2.\end{aligned}$$

The desired sum then telescopes:

$$\begin{aligned}\sum_{k=1}^{n}(5F_{2^k}^4+3F_{2^k}^2) &= \sum_{k=1}^{n}(F_{2^{k+1}}^2-F_{2^k}^2)\\ &= F_{2^{n+1}}^2-1\\ &= (F_{2^{n+1}}-1)(F_{2^{n+1}}+1).\end{aligned}$$

Also solved by Thomas Achammer, Illia Antypenko (high school student), Michel Bataille, Nandan Sai Dasireddy, Steve Edwards, Dmitry Fleischman, G. C. Greubel, Haydn Gwyn (undergraduate), Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), J. N. Senadherra, Albert Stadler, Andrés Ventas, Dan Weiner, and the proposer.