ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-889 Proposed by Kapil Kumar Gurjar, Uttar Pradesh, India

Let $\lfloor x \rfloor$ and $\lceil x \rceil$ be the integer part and the ceiling of the real number x, respectively. Then for each integer $m \geq 2$ and $n \geq 2^{m+1}+1$, we have

(1) $\lfloor \sqrt[2^m]{F_n} \rfloor = \lfloor \sqrt[2^m]{F_{n-2^m}} + \sqrt[2^m]{F_{n-2^{m+1}}} \rfloor$.

(2) The equality (1) above holds with F replaced by L. It also holds with either F or L and with the floor function replaced by the ceiling function.

(3) $\sqrt[2^m]{F_n} - \sqrt[2^m]{F_{n-2^m}} - \sqrt[2^m]{F_{n-2^{m+1}}} > 1/10^{2^m}$ and the same inequality holds with F replaced by L.

H-890 Proposed by Ryan Zielinski, Clifton, NJ

Prove that, for nonnegative integers m,

$$\frac{1}{5^m}\sum_{j=0}^{m}\binom{2m+1}{m-j}L_{2j+1}(2j+1) = 2m+1.$$

H-891 Proposed by Robert Frontczak, Stuttgart, Germany

Prove that for all $n \geq 1$,

$$\frac{F_nF_{10n} - F_{2n}F_{5n}}{5F_n^2} \equiv 0 \pmod{10}.$$

H-892 Proposed by Toyesh Prakash Sharma, Agra, India

Evaluate the following limit

$$\lim_{n\to\infty}\frac{L_{n-1}+L_{n+1}}{n\sqrt[n]{n!}}\sqrt[n+1]{1+\sum_{m=1}^{n}\int_0^1 m^2\ln^{m-1}(1/x)dx}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{F_{n-1}}\right).$$

H-893 Proposed by Seán M. Stewart, Thuwal, Saudi Arabia

Prove

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n F_n}{2^n} = \log\left(\frac{5}{4}\right) + \frac{6}{\sqrt{5}} \log \alpha.$$

Here, $\overline{H}_n$ is the nth skew harmonic number defined by $\sum_{k=1}^{n} (-1)^{k+1}/k$.

H-894 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For any integers r and $n \geq 1$ prove that

(i) $\displaystyle\sum_{k=1}^{n} F_k F_{k+r} F_{2k+r} = F_n F_{n+1} F_{n+r} F_{n+r+1};$

(ii) $\displaystyle\sum_{k=1}^{n} F_k^5 F_{k+1}^5 F_{2k+1} = \frac{1}{8}(F_n F_{n+1} F_{n+2})^4.$

SOLUTIONS

A double limit with a function satisfying a functional equation

H-854 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania (Vol. 58, No. 2, May 2020)

Compute

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{\frac{L_n}{(x+1)L_{n+1}}} - (f(x))^{\frac{L_n}{xL_{n+1}}}\right) x^{\frac{L_{n-1}}{L_{n+1}}}\right),$$

where $f : \mathbb{R}_+^* \mapsto \mathbb{R}_+^*$ is a function that satisfies $\lim_{x\to\infty} f(x+1)/(xf(x)) = a \in \mathbb{R}_+^*$.

Solution by the proposers

We denote $u_n = L_n/L_{n+1}$ and we have $\lim\limits_{n\to\infty} u_n = 1/\alpha$. We also have

$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)^{\frac{1}{x}}}{x} &= \lim_{n\to\infty} \frac{f(n)^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{f(n)}{n^n}} = \lim_{n\to\infty} \frac{f(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{f(n)} \\ &= \lim_{n\to\infty} \frac{f(n+1)}{nf(n)} \left(\frac{n}{n+1}\right)^{n+1} = \frac{a}{e}.\end{aligned}$$

We denote

$$v_n(x) = \frac{f(x+1)^{\frac{L_n}{(x+1)L_{n+1}}}}{f(x)^{\frac{L_n}{xL_{n+1}}}} = \left(\frac{f(x+1)^{\frac{1}{x+1}}}{f(x)^{\frac{1}{x}}}\right)^{u_n}.$$

We have $\lim_{x\to\infty} v_n(x) = 1$, so

$$\lim_{x\to\infty} \frac{v_n(x) - 1}{\ln v_n(x)} = 1$$

and

$$\begin{aligned}\lim_{x\to\infty} v_n(x)^x &= \lim_{x\to\infty} \left(\frac{f(x+1)}{f(x)} \cdot \frac{1}{f(x+1)^{\frac{1}{x+1}}}\right)^{u_n} \\ &= \lim_{x\to\infty} \left(\frac{f(x+1)}{xf(x)}\right)^{u_n} \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{u_n} \lim_{x\to\infty} \left(\frac{x+1}{f(x+1)^{\frac{1}{x+1}}}\right)^{u_n} \\ &= a^{u_n} \cdot 1 \cdot \left(\frac{e}{a}\right)^{u_n} = e^{u_n}.\end{aligned}$$

Hence,

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\left(f(x+1)\right)^{\frac{L_n}{(x+1)L_{n+1}}}-\left(f(x)\right)^{\frac{L_n}{xL_{n+1}}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)\\
&=\lim_{n\to\infty}\lim_{x\to\infty}\left(f(x+1)^{\frac{u_n}{x+1}}-f(x)^{\frac{u_n}{x}}\right)x^{1-u_n}=\lim_{n\to\infty}\lim_{x\to\infty}f(x)^{\frac{u_n}{x}}(v_n(x)-1)x^{1-u_n}\\
&=\lim_{n\to\infty}\lim_{x\to\infty}f(x)^{\frac{u_n}{x}}\left(\frac{v_n(x)-1}{\ln v_n(x)}\right)x^{1-u_n}\ln v_n(x)\\
&=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\frac{f(x)^{\frac{1}{x}}}{x}\right)^{u_n}\cdot\lim_{x\to\infty}\left(\frac{v_n(x)-1}{\ln v_n(x)}\right)\cdot\lim_{x\to\infty}\ln(v_n(x)^x)\right)\\
&=\lim_{n\to\infty}\left(\frac{a}{e}\right)^{u_n}\cdot 1\cdot\ln e^{u_n}=\frac{1}{\alpha}\left(\frac{a}{e}\right)^{\frac{1}{\alpha}}.
\end{aligned}
$$

Also solved by Dmitry Fleischman and Albert Stadler.

The generating function of the product of the Fibonacci and Tribonacci numbers

H-855 Proposed by Robert Frontczak, Stuttgart, Germany
(Vol. 58, No. 2, May 2020)

Let $(T_n)_{n\geq 0}$ be the sequence of Tribonacci numbers given by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$. Define the functions

$$G_{FT}(z)=\sum_{n=0}^{\infty}F_nT_nz^n \quad\text{and}\quad G_{LT}(z)=\sum_{n=0}^{\infty}L_nT_nz^n.$$

Show that for $k \geq 1$, we have

$$G_{FT}(2^{-2k})=\frac{2^{4k}(2^{6k}-2^{2k}-1)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}-2^{4k+1}+2^{2k}-1}$$

and

$$G_{LT}(2^{-2k})=\frac{2^{4k}(2^{6k}+2^{4k+1}+2^{2k}+3)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}-2^{4k+1}+2^{2k}-1}.$$

Solution by Ángel Plaza, Gran Canaria, Spain

Because the generating function for the Tribonacci numbers is

$$T(z)=\sum_{n=0}^{\infty}T_nz^n=\frac{z}{1-z-z^2-z^3},$$

it follows by the Binet's formulas that

$$G_{FT}(z)=\frac{T(\alpha z)-T(\beta z)}{\sqrt{5}} \quad\text{and}\quad G_{LT}(z)=T(\alpha z)+T(\beta z)$$

from where

$$G_{FT}(z) = \frac{z\left(z^3+z^2-1\right)}{z^6-z^5+2z^4+5z^3+4z^2+z-1} \quad \text{and}$$
$$G_{LT}(z) = \frac{z\left(3z^3+z^2+2z+1\right)}{z^6-z^5+2z^4+5z^3+4z^2+z-1}, \quad \text{so}$$
$$G_{FT}(2^{-2k}) = \frac{2^{4k}(2^{6k}-2^{2k}-1)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}+2^{2k}-1} \quad \text{and}$$
$$G_{LT}(2^{-2k}) = \frac{2^{4k}(2^{6k}+2^{4k+1}+2^{2k}+3)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}+2^{2k}-1}.$$

Also solved by Brian Bradie, Dmitry Fleischman, Kapil Kumar Gurjar, Raphael Schumacher, Jason L. Smith, Albert Stadler, and the proposer.

The sum of a series with Fibonacci and triangular numbers

H-856 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 58, No. 2, May 2020)

Let T_n denote the nth triangular number; i.e., $T_n = n(n+1)/2$. Show that

$$\sum_{n=0}^{\infty} T_n \cdot \frac{F_n}{2^{n+2}} = F_7 \quad \text{and} \quad \sum_{n=0}^{\infty} T_n \cdot \frac{L_n}{2^{n+2}} = L_7.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan

By the identity

$$\frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^{n+1} = \frac{d^2}{dx^2}\frac{x}{1-x} \quad \text{for} \quad |x|<1,$$

we have

$$\sum_{n=0}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3}.$$

Therefore, we have

$$\sum_{n=0}^{\infty} T_n \frac{x^n}{4} = \frac{x}{4(1-x)^3}.$$

Evaluating at $x=\alpha/2$ and $x=\beta/2$ the above identity we get

$$\sum_{n=0}^{\infty} T_n \cdot \frac{\alpha^n}{2^{n+2}} = \frac{29+13\sqrt{5}}{2} \quad \text{and} \quad \sum_{n=0}^{\infty} T_n \cdot \frac{\beta^n}{2^{n+2}} = \frac{29-13\sqrt{5}}{2}.$$

We obtain the desired identities by using the Binet formulas for F_n and L_n.

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook, Dmitry Fleischman, Kapil Kumar Gurjar, Ángel Plaza, Diego Rattaggi, Henri Ricardo, Raphael Schumacher, J. N. Senadheere, Jason L. Smith, Albert Stadler, David Terr, and the proposer.

Note: The reader is referred to the retraction notice from the end of the Advanced Problem Section of **Vol. 58, No. 3, August 2020** issue.

Fibonacci numbers and signed sums of products of Tribonacci numbers

H-857 Proposed by T. Goy, Ivano-Frankivsk, Ukraine
(Vol. 58, No. 2, May 2020)

Let T_n be the nth Tribonacci number given by $T_0 = T_1 = 0$, $T_2 = 1$, and for $n \geq 3$, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. For all $n \geq 2$, prove that

$$F_{n-2} = \sum_{i=1}^{n-1} (-1)^{i-1} \sum_{s_1+\cdots+s_i=n} T_{s_1-1} T_{s_2-1} \cdots T_{s_i-1}.$$

Solution by Tom Edgar

Let F_n be the Fibonacci sequence and T_n be the Tribonacci sequence. Let $G_F(x)$ be the generating function for F_n and $G_T(x)$ be the generating function for T_n. Then,

$$G_F(x) = \frac{x}{1-x-x^2} \qquad \text{and} \qquad G_T(x) = \frac{x^2}{1-x-x^2-x^3};$$

these results come from standard solutions to linear recurrence relations (or see A000045 and A000073 in OEIS). Consequently, we have

$$\sum_{i=1}^{\infty} T_{i-1} x^i = x G_T(x) = \frac{x^3}{1-x-x^2-x^3}. \tag{1}$$

Next, as shown in Concrete Mathematics ([1], p. 355), the generating function for the sequence

$$A_{n,i} = (-1)^{i-1} \sum_{s_1+s_2+\cdots+s_i=n} T_{s_1-1} T_{s_2-1} \cdots T_{s_i-1}$$

is given by $(-1)^{i-1}(xG_T(x))^i$. Therefore, the generating function for

$$\sum_{i \geq 1} (-1)^{i-1} \sum_{s_1+s_2+\cdots+s_i=n} T_{s_1-1} T_{s_2-1} \cdots T_{s_i-1}$$

is

$$xG_T(x) - (xG_T(x))^2 + (xG_T(x))^3 - (xG_T(x))^4 + \cdots = \sum_{i \geq 1} (-1)^{i-1} (xG_T(x))^i,$$

again, see Concrete Mathematics ([1], p. 356).

Applying the formal geometric series identity yields

$$xG_T(x) - (xG_T(x))^2 + (xG_T(x))^3 - (xG_T(x))^4 + \cdots = \sum_{i \geq 1} (-1)^{i-1} (xG_T(x))^i = \frac{xG_T(x)}{1 + xG_T(x)}.$$

According to equation (1), we then see that

$$\frac{xG_T(x)}{1+xG_T(x)} = \frac{\frac{x^3}{1-x-x^2-x^3}}{1 + \frac{x^3}{1-x-x^2-x^3}} = \frac{x^3}{1-x-x^2} = x^2 G_F(x).$$

Thus, we have that $x^2 G_F(x)$ is the generating function for

$$\sum_{i \geq 1} \sum_{s_1+s_2+\cdots+s_i=n} T_{s_1-1} T_{s_2-1} \cdots T_{s_i-1},$$

and the coefficient of x^n in $x^2 G_F(x)$ is F_{n-2}. Thus,

$$F_{n-2} = \sum_{i\geq 1} \sum_{s_1+s_2+\cdots+s_i=n} T_{s_1-1}T_{s_2-1}\cdots T_{s_i-1}.$$

Finally, we note that the sum on the right side is zero when $i > n$ (because we cannot have more than n positive integers sum to n), and when $i = n$ (because in this case each $s_k = 1$ and the summand is simply the product of n copies of $T_0 = 0$. Hence, we conclude that

$$F_{n-2} = \sum_{i=1}^{n-1} \sum_{s_1+s_2+\cdots+s_i=n} T_{s_1-1}T_{s_2-1}\cdots T_{s_i-1}.$$

as required.

Reference

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.

Also solved by Dmitry Fleischman, Raphael Schumacher, Albert Stadler, and the proposer.

A perfect square

H-858 Proposed by Muneer Jebreel Karama, Hebron, Palestine (Vol. 58, No. 3, August 2020)

Show that

$$\frac{1}{2}\left((2F_nF_{n+1})^8 + (F_{n-1}F_{n+2})^8 + F_{2n+1}^8\right)$$

is a perfect square for all $n \geq 0$.

Solution by Hideyuki Ohtsuka, Saitama, Japan

The desired expression is a square because

$$F_{n-1}F_{n+2} = F_{n+1}^2 - F_n^2, \quad F_{2n+1} = F_n^2 + F_{n+1}^2 \quad \text{(see [1](11), (12))}$$

and

$$\frac{1}{2}((2ab)^8 + (a^2-b^2)^8 + (a^2+b^2)^8) = (a^8 + 14a^4b^4 + b^8)^2.$$

Reference

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Also solved by Brian Bradie, Dmitry Fleischman, Wei-Kai Lai, Ángel Plaza, Raphael Schumacher, Jason L. Smith, Albert Stadler, and the proposer.

A series with Fibonacci numbers and values of the Riemann zeta function

H-859 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 58, No. 3, August 2020)

Prove that

$$\sum_{n\geq 1} \zeta(2n+1)\frac{F_{2n}}{5^n} = \frac{1}{2},$$

where $\zeta(k) = \sum_{n\geq 1} 1/n^k$ for $k \geq 2$ is the Riemann zeta function.

Solution by Raphael Schumacher, ETH, Zurich, Switzerland

We can calculate that

$$\begin{aligned}\sum_{n=1}^{\infty}\zeta(2n+1)\frac{F_{2n}}{5^n} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\frac{F_{2n}}{5^n} = \frac{1}{\sqrt{5}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{n=1}^{\infty}\left(\frac{\alpha^2}{5k^2}\right)^n - \sum_{n=1}^{\infty}\left(\frac{\beta^2}{5k^2}\right)^n\right)\\
&= \frac{1}{\sqrt{5}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\frac{\alpha^2}{5k^2}}{1-\frac{\alpha^2}{5k^2}} - \frac{\frac{\beta^2}{5k^2}}{1-\frac{\beta^2}{5k^2}}\right) = \frac{1}{\sqrt{5}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\alpha^2}{5k^2-\alpha^2} - \frac{\beta^2}{5k^2-\beta^2}\right)\\
&= \frac{1}{\sqrt{5}}\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{5\sqrt{5}k^2}{25k^4-15k^2+1} = \frac{1}{\sqrt{5}}\sum_{k=1}^{\infty}\frac{5\sqrt{5}k}{25k^4-15k^2+1}\\
&= 5\sum_{k=1}^{\infty}\frac{1}{10}\left(\frac{1}{5k^2-5k+1} - \frac{1}{5k^2+5k+1}\right)\\
&= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{5k^2-5k+1} - \frac{1}{5(k+1)^2-5(k+1)+1}\right) = \frac{1}{2}\cdot(1) = \frac{1}{2}.\end{aligned}$$

In the second step, we could interchange the two summations by absolute convergence.
In the above calculation, we used the identity

$$\frac{\alpha^2}{5k^2-\alpha^2} - \frac{\beta^2}{5k^2-\beta^2} = \frac{5\left(\alpha^2-\beta^2\right)k^2}{25k^4-5\left(\alpha^2+\beta^2\right)k^2+(\alpha\beta)^2} = \frac{5\sqrt{5}k^2}{25k^4-15k^2+1},$$

which follows from the identities $\alpha\beta = -1$, $\alpha^2-\beta^2 = \sqrt{5}$, and $\alpha^2+\beta^2 = 3$.

Also solved by Khristo Boyadzhiev, Brian Bradie, Alejandro Cardona Castrillón, Dmitry Fleischman, J. N. Senadheera, Albert Stadler, Seán M. Stewart, David Terr, and the proposer.