ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-817 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For $n \geq 1$, find closed form expressions for the sums

(i) $\displaystyle\sum_{k=1}^{n} F_{2^k} F_{2^k-1} F_{2^{k+1}-1} \cdots F_{2^n-1}$;

(ii) $\displaystyle\sum_{k=1}^{n} F_{2^k-3} L_{2^k-1} L_{2^{k+1}-1} \cdots L_{2^n-1}$;

(iii) $\displaystyle\sum_{k=1}^{n} (-1)^k F_{2^k} L_{2^k-1} L_{2^{k+1}-1} \cdots L_{2^n-1}$;

(iv) $\displaystyle\sum_{k=1}^{n} (-1)^k G_{2^k+k} L_{2^k-1} L_{2^{k+1}-1} \cdots L_{2^n-1}$,

where $\{G_n\}_{n\geq 1}$ satisfies $G_{n+2} = G_{n+1} + G_n$ for $n \geq 1$ with arbitrary G_1 and G_2.

H-818 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Determine

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+4}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2} F_{n+3} F_{n+4}}.$$

H-819 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and odd function and $g : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ be a continuous function such that $g(1/x) = -g(x)$ for all $x \in \mathbb{R}_+^*$. Compute

$$\int_{-\beta}^{\alpha} \frac{dx}{(1+x^2)(1+e^{(f\circ g)(x)})},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

H-820 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania

If $a, b, c \in \mathbb{R}_+$, compute

$$\lim_{n\to\infty} \frac{\left(\sqrt[n+1]{(2n+1)!!F_{n+1}^b}\right)^{a+1} - \left(\sqrt[n]{(2n-1)!!F_n^b}\right)^{a+1}}{\left(\sqrt[n]{n!L_n^c}\right)^a}.$$

SOLUTIONS

Closed forms for sums of series involving reciprocals of shifted Fibonacci squares

H-783 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 54, No. 1, February 2016)

Prove that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_n^2+1} = \frac{-3+5\sqrt{5}}{6}$;

(ii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{F_n^2-1} = \frac{43-15\sqrt{5}}{18}$;

(iii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{F_n^4-1} = \frac{35-15\sqrt{5}}{18}$.

Solution by Ángel Plaza

(i) We will show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{F_{2n}^2+1} = \alpha = \frac{1+\sqrt{5}}{2}$, and that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}^2+1} = \frac{\sqrt{5}}{3}$. These two series are consequences of the following two identities that may be proved by induction:

$$\sum_{n=0}^{m} \frac{1}{F_{2n}^2+1} = \frac{F_{2m+2}}{F_{2m+1}}, \qquad \sum_{n=0}^{m} \frac{1}{F_{2n+1}^2+1} = \frac{F_{4m+4}/3}{F_{2m+1}F_{2m+3}}.$$

Therefore, the sum proposed in (i) is

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2+1} = \sum_{n=1}^{\infty} \frac{1}{F_{2n}^2+1} + \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}^2+1} = \alpha - 1 + \frac{\sqrt{5}}{3} = \frac{-3+5\sqrt{5}}{6}.$$

□

(ii) Since $\dfrac{1}{F_n^4-1}=\dfrac{1/2}{F_n^2-1}-\dfrac{1/2}{F_n^2+1}$, then

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{1}{F_n^2-1} &= 2\sum_{n=3}^{\infty}\frac{1}{F_n^4-1}+\sum_{n=3}^{\infty}\frac{1}{F_n^2+1}\\ &= \frac{35-15\sqrt{5}}{18}+\frac{-3+5\sqrt{5}}{6}-1\\ &= \frac{43-15\sqrt{5}}{18}\end{aligned}$$

where we have used the sum given in (iii), which is proved below. □

(iii) First, note that $F_n^4-1=F_{n-2}F_{n-1}F_{n+1}F_{n+2}$ and that $F_n=\dfrac{F_{n+2}+F_{n-2}}{3}$. Therefore,

$$\frac{1}{F_n^4-1}=\frac{1/3}{F_{n-2}F_{n-1}F_nF_{n+1}}+\frac{1/3}{F_{n-1}F_nF_{n+1}F_{n+2}}.$$

Taking into account the following relation equation (24) in [1]:

$$\sum_{i=1}^{n-1}\frac{1}{F_iF_{i+1}F_{i+2}F_{i+3}}=\frac{7}{4}-\frac{1}{2}\left(\frac{F_{n-1}}{F_n}+\frac{3F_n}{F_{n+1}}+\frac{F_{n+1}}{F_{n+2}}\right)$$

it is deduced that

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{1/3}{F_{n-2}F_{n-1}F_nF_{n+1}} &= \frac{1}{3}\left(\frac{7}{4}-\frac{5}{2\alpha}\right),\\ \sum_{n=3}^{\infty}\frac{1/3}{F_{n-1}F_nF_{n+1}F_{n+2}} &= \frac{1}{3}\left(\frac{7}{4}-\frac{5}{2\alpha}-\frac{1}{6}\right),\end{aligned}$$

from where the sum (iii) follows.

[1] R. S. Melham, *Finite sums that involve reciprocal of products of generalized Fibonacci numbers*, Integers, **13.4** (2013), A40.

Also solved by Brian Bradie, Dmitry Fleischman, and the proposer.

A pair of identities for π

H-784 Proposed by Gleb Glebov, Simon Fraser University, Canada (Vol. 54, No. 1, February 2016)

Prove that

(i) $$\sum_{k=1}^{\infty}\left[\frac{1}{24k+11}-\frac{1}{24k-11}+\frac{1}{24k+1}-\frac{1}{24k-1}\right]=\frac{\pi(\sqrt{6}+\sqrt{2})}{12}-\frac{12}{11};$$

(ii) $$\sum_{k=1}^{\infty}\left[\frac{1}{24k+7}-\frac{1}{24k-7}+\frac{1}{24k+5}-\frac{1}{24k-5}\right]=\frac{\pi(\sqrt{6}-\sqrt{2})}{12}-\frac{12}{35}.$$

Solution by Hideyuki Ohtsuka

It is known that

$$\pi x\cot\pi x=1-\sum_{k=1}^{\infty}\frac{2x^2}{k^2-x^2}.$$

From the above identity, we have

$$\sum_{k=1}^{\infty} \frac{1}{(24k)^2 - (24x)^2} = \frac{1 - \pi x \cot \pi x}{2(24x)^2}. \tag{1}$$

(i) Note that

$$\cot \frac{11\pi}{24} = -2 + \sqrt{2} - \sqrt{3} + \sqrt{6} \qquad \text{and} \qquad \cot \frac{\pi}{24} = 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}.$$

We have

$$\begin{aligned} LHS &= -22 \sum_{k=1}^{\infty} \frac{1}{(24k)^2 - 11^2} - 2 \sum_{k=1}^{\infty} \frac{1}{(24k)^2 - 1^2} \\ &= -\frac{22}{2 \times 11^2} \left(1 - \frac{11\pi}{24} \cot \frac{11\pi}{24}\right) - \frac{2}{2 \times 1^2} \left(1 - \frac{\pi}{24} \cot \frac{\pi}{24}\right) \\ &= -\frac{1}{11} + \frac{\pi}{24}(-2 + \sqrt{2} - \sqrt{3} + \sqrt{6}) - 1 + \frac{\pi}{24}(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}) \\ &= RHS. \end{aligned}$$

. (ii) Note that

$$\cot \frac{7\pi}{24} = -2 - \sqrt{2} + \sqrt{3} + \sqrt{6} \qquad \text{and} \qquad \cot \frac{5\pi}{24} = 2 - \sqrt{2} - \sqrt{3} + \sqrt{6}.$$

We have

$$\begin{aligned} LHS &= -14 \sum_{k=1}^{\infty} \frac{1}{(24k)^2 - 7^2} - 10 \sum_{k=1}^{\infty} \frac{1}{(24k)^2 - 5^2} \\ &= -\frac{14}{2 \times 7^2} \left(1 - \frac{7\pi}{24} \cot \frac{7\pi}{24}\right) - \frac{10}{2 \times 5^2} \left(1 - \frac{5\pi}{24} \cot \frac{5\pi}{24}\right) \\ &= -\frac{1}{7} + \frac{\pi}{24}(-2 - \sqrt{2} + \sqrt{3} + \sqrt{6}) - \frac{1}{5} + \frac{\pi}{24}(2 - \sqrt{2} - \sqrt{3} + \sqrt{6}) \\ &= RHS. \end{aligned}$$

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Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, David Terr, Nicuşor Zlota, and the proposer.

Sums of Fibonomial coefficients

H-785 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 54, No. 1, February 2016)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $m \geq n \geq 1$, find closed forms expressions for the sums

(i) $\displaystyle\sum_{k=0}^{n} F_{2k}\binom{2n}{n+k}_F\binom{2m}{m+k}_F$;

(ii) $\displaystyle\sum_{k=0}^{n} F_{2k}\binom{2n}{n+k}_F^{-1}\binom{2m}{m+k}_F^{-1}$.

Solution by the proposer

It is known that

$$F_{a+r}F_{b+r} - (-1)^r F_a F_b = F_{a+b+r}F_r \qquad \text{(see [1](20}a\text{))}. \tag{2}$$

Putting $a = s-k$, $b = t-k$, and $r = 2k$ in the above identity, we have

$$F_{s+k}F_{t+k} - F_{s-k}F_{t-k} = F_{s+t}F_{2k}. \tag{3}$$

(i) We have

$$\begin{aligned}
&\binom{2n-1}{n+k-1}_F\binom{2m-1}{m+k-1}_F - \binom{2n-1}{n+k}_F\binom{2m-1}{m+k}_F \\
= \ & \frac{F_{n+k}}{F_{2n}}\binom{2n}{n+k}_F\frac{F_{m+k}}{F_{2m}}\binom{2m}{m+k}_F - \frac{F_{n-k}}{F_{2n}}\binom{2n}{n+k}_F\frac{F_{m-k}}{F_{2m}}\binom{2m}{m+k}_F \\
= \ & \frac{F_{n+k}F_{m+k} - F_{n-k}F_{m-k}}{F_{2n}F_{2m}}\binom{2n}{n+k}_F\binom{2m}{m+k}_F \\
= \ & \frac{F_{n+m}F_{2k}}{F_{2n}F_{2m}}\binom{2n}{n+k}_F\binom{2m}{m+k}_F \qquad \text{(by (3))}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{k=0}^{n} F_{2k}\binom{2n}{n+k}_F\binom{2m}{m+k}_F \\
= \ & \frac{F_{2n}F_{2m}}{F_{n+m}}\sum_{k=0}^{n}\left[\binom{2n-1}{n+k-1}_F\binom{2m-1}{m+k-1}_F - \binom{2n-1}{n+k}_F\binom{2m-1}{m+k}_F\right] \\
= \ & \frac{F_{2n}F_{2m}}{F_{n+m}}\left[\binom{2n-1}{n-1}_F\binom{2m-1}{m-1}_F - \binom{2n-1}{2n}_F\binom{2m-1}{m+n}_F\right] \\
= \ & \frac{F_{2n}F_{2m}}{F_{n+m}} \times \frac{F_n}{F_{2n}}\binom{2n}{n}_F\frac{F_m}{F_{2m}}\binom{2m}{m}_F = \frac{F_nF_m}{F_{n+m}}\binom{2n}{n}_F\binom{2m}{m}_F.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
&\binom{2n+1}{n+k+1}_F^{-1}\binom{2m+1}{m+k+1}_F^{-1}-\binom{2n+1}{n+k}_F^{-1}\binom{2m+1}{m+k}_F^{-1}\\
=\ &\frac{F_{n+k+1}}{F_{2n+1}}\binom{2n}{n+k}_F^{-1}\frac{F_{m+k+1}}{F_{2m+1}}\binom{2m}{m+k}_F^{-1}-\frac{F_{n-k+1}}{F_{2n+1}}\binom{2n}{n+k}_F^{-1}\frac{F_{m-k+1}}{F_{2m+1}}\binom{2m}{m+k}_F^{-1}\\
=\ &\frac{F_{n+k+1}F_{m+k+1}-F_{n+1-k}F_{m+1-k}}{F_{2n+1}F_{2m+1}}\binom{2n}{n+k}_F^{-1}\binom{2m}{m+k}_F^{-1}\\
=\ &\frac{F_{n+m+2}F_{2k}}{F_{2n+1}F_{2m+1}}\binom{2n}{n+k}_F^{-1}\binom{2m}{m+k}_F^{-1}\qquad\text{(by (3))}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{k=0}^{n}F_{2k}\binom{2n}{n+k}_F^{-1}\binom{2m}{m+k}_F^{-1}\\
=\ &\frac{F_{2n+1}F_{2m+1}}{F_{n+m+2}}\sum_{k=0}^{n}\left[\binom{2n+1}{n+k+1}_F^{-1}\binom{2m+1}{m+k+1}_F^{-1}-\binom{2n+1}{n+k}_F^{-1}\binom{2m+1}{m+k}_F^{-1}\right]\\
=\ &\frac{F_{2n+1}F_{2m+1}}{F_{n+m+2}}\left[\binom{2n+1}{2n+1}_F^{-1}\binom{2m+1}{m+n+1}_F^{-1}-\binom{2n+1}{n}_F^{-1}\binom{2m+1}{m}_F^{-1}\right]\\
=\ &\frac{F_{2n+1}F_{2m+1}}{F_{n+m+2}}\left[\frac{F_{m+n+1}}{F_{2m+1}}\binom{2m}{m+n}_F^{-1}-\frac{F_{n+1}}{F_{2n+1}}\binom{2n}{n}_F^{-1}\frac{F_{m+1}}{F_{2m+1}}\binom{2m}{m}_F^{-1}\right]\\
=\ &\frac{F_{2n+1}F_{n+m+1}}{F_{n+m+2}}\binom{2m}{n+m}_F^{-1}-\frac{F_{n+1}F_{m+1}}{F_{n+m+2}}\binom{2n}{n}_F^{-1}\binom{2m}{m}_F^{-1}.
\end{aligned}$$

Note: Similarly, for positive integers n and r we obtain

$$\sum_{k=0}^{n}F_{2k}\binom{n}{r+k}_F\binom{n}{r-k}_F=\frac{F_rF_{n-r}}{F_n}\binom{n}{r}_F^2.$$

[1] S. Vajda, *Fibonacci and Lucas numbers and the golden section*, Dover, 2008.

The area of a Fibonacci polygon

H-786 Proposed by Atara Shriki, Oranim College of Education (Vol. 54, No. 1, February 2016)

Assume that the consecutive numbers in the Fibonacci sequence are the coordinates of a polygon's vertices in the Cartesian coordinate system, counterclockwise:

$$A_1(F_1,F_2);\ A_2(F_3,F_4);\ A_3(F_5,F_6);\ A_4(F_7,F_8);\ \ldots;\ A_n(F_{2n-1},F_{2n}).$$

What is the area of such a polygon?

Solution by Virginia Johnson

One formula for area bounded by a polygon with coordinates with vertices at $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $\ldots$, $P_n(x_n, y_n)$ is the so called shoelace formula or surveyor's formula, given by the absolute value of

$$\frac{1}{2}(x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1 - y_1x_2 - y_2x_3 - \cdots - y_{n-1}x_n - y_nx_1)$$

See reference [1].

Taking the vertices in counterclockwise order, the area of the polygons is

$$A = \frac{1}{2}\Big(F_1F_{2n} + F_{2n-1}F_{2n-2} + F_{2n-3}F_{2n-4} + \cdots + F_5F_4 + F_3F_2 \\ - F_2F_{2n-1} - F_{2n}F_{2n-3} - F_{2n-2}F_{2n-5} - \cdots - F_6F_3 - F_4F_1\Big)$$

Reordering the terms, we have

$$A = \frac{1}{2}\Big((F_1F_{2n} - F_2F_{2n-1}) + (F_{2n-1}F_{2n-2} - F_{2n}F_{2n-3}) \\ + (F_{2n-3}F_{2n-4} - F_{2n-2}F_{2n-5}) + \cdots + (F_5F_4 - F_6F_3) + (F_3F_2 - F_4F_1)\Big). \tag{4}$$

Note that after the first pair, each of the subsequent $(n-1)$ pairs have the form $F_{2j-1}F_{2j-2} - F_{2j}F_{2j-3}$. Using an identity from Everman, et al. [2]:

$$F_{n+k}F_{n+h} - F_nF_{n+h+k} = (-1)^nF_hF_k,$$

we have that equation (4) reduces to

$$A = \frac{F_1F_{2n} - F_2F_{2n-1} - 1(n-1)}{2} = \frac{F_{2n} - F_{2n-1} - n + 1}{2} = \frac{F_{2n-2} - n + 1}{2}.$$

Therefore, the area of the polygon is $\dfrac{F_{2n-2} - n + 1}{2}$.

[1] B. Braden, *The surveyor's area formula*, The College Mathematics Journal, **17.4** (1986), 326–337.

[2] D. Everman, A. Danese, K. Venkannayah, and E. Scheuer, *Elementary problems and solutions: Some properties of Fibonacci numbers*, The American Mathematical Monthly, **67.7** (1960), 694.

Also solved by Harris Kwong, Ángel Plaza, and the proposer.

Errata: In the statement of **H-815**, the condition "$p > 5$" must be added.

Withdrawals: Problem **H-816** is withdrawn as being a particular case of **B-1173**.