ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-850 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For integers m, n, r, and s, let

$$\overrightarrow{AB} = (F_m, F_{m+r}, F_{m+s}) \quad \text{and} \quad \overrightarrow{AC} = (F_n, F_{n+r}, F_{n+s}).$$

Prove that the area of the triangle ABC is

$$\frac{1}{2}\sqrt{F_r^2 + F_s^2 + F_{r-s}^2}|F_{n-m}|.$$

H-851 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania and Neculai Stanciu, Buzău, Romania

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of positive real numbers such that $\lim_{n\to\infty} a_{n+1}/(n^r a_n) = a \in \mathbb{R}_+^*$ and $\lim_{n\to\infty} b_{n+1}/(n^s b_n) = b \in \mathbb{R}_+^*$, where $r, s \in \mathbb{R}_+$. Compute

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{F_{n+1}}}{n^{r+s}} - \frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{F_n}}{(n+1)^{r+s}}\right)\sqrt[n]{b_n}.$$

H-852 Proposed by Robert Frontczak, Stuttgart, Germany

Let $(B_n)_{n\geq 0}$ denote the Bernoulli numbers. Show that for all $r \geq 1$ and $n \geq 3$,

$$\sum_{k=0}^{n}\binom{n}{k}F_{rk}L_{r(n-k)}B_kB_{n-k} = \begin{cases}(1-n)B_nB_{rn}, & n \text{ even};\\ -nB_{n-1}F_{rn}, & n \text{ odd}.\end{cases}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(2^{1-k}-1)(2^{1-(n-k)-1}-1)F_{rk}L_{r(n-k)}B_kB_{n-k} = \begin{cases}(1-n)B_nB_{rn}, & n \text{ even};\\ 0, & n \text{ odd}.\end{cases}$$

H-853 **Proposed by Ángel Plaza and Sergio Falcón, Gran Canaria, Spain**

Let L_n be the nth Lucas number given by the recurrence $L_{n+2} = kL_{n+1} + L_n$ for all $n \geq 0$ with $L_0 = 2$ and $L_1 = k$. Prove that

(i) $\displaystyle\sum_{j=1}^{n} \frac{L_j^2}{L_j + 1} \geq \frac{(L_n + L_{n+1} - k - 2)^2}{k\sqrt{kn(L_n + L_{n+1} + k(n-1) - 2)}}$;

(ii) $\displaystyle\sum_{j=1}^{n} \frac{L_j^4}{L_j^2 + 1} \geq \frac{(L_{2n+1} + k((-1)^n - 2))^2}{k\sqrt{kn(L_{2n+1} + k(n - 2 + (-1)^n))}}$.

SOLUTIONS

Closed form expressions for some sums of products

H-817 **Proposed by Hideyuki Ohtsuka, Saitama, Japan**
(Vol. 56, No. 1, February 2018)

For $n \geq 1$ find closed form expressions for the sums

(i) $\displaystyle\sum_{k=1}^{n} F_{2^k} F_{2^k-1} F_{2^{k+1}-1} \cdots F_{2^n-1}$;

(ii) $\displaystyle\sum_{k=1}^{n} F_{2^k-3} L_{2^k-1} L_{2^{k+1}-1} \cdots L_{2^n-1}$;

(iii) $\displaystyle\sum_{k=1}^{n} (-1)^k F_{2^k} L_{2^k-1} L_{2^{k+1}-1} \cdots L_{2^n-1}$;

(iv) $\displaystyle\sum_{k=1}^{n} (-1)^k G_{2^k+k} L_{2^k-1} L_{2^{k+1}-1} \cdots L_{2^n-1}$,

where $\{G_n\}_{n\geq 1}$ satisfies $G_{n+2} = G_{n+1} + G_n$ for $n \geq 1$ with arbitrary G_1 and G_2.

Solution by the proposer

First, we show the following lemma.

Lemma. *Let $a_0 = 0$ and $b_n \neq 0$ for $n \geq 1$. For $n \geq 1$, we have*

$$\sum_{k=1}^{n} \left(\frac{a_k}{b_k} - a_{k-1} \right) \prod_{j=k}^{n} b_j = a_n.$$

Proof of the lemma. For $n = 1$, we have

$$LS = \left(\frac{a_1}{b_1} - a_0 \right) b_1 = a_1 = RS.$$

For $n \geq 2$, we have

$$\sum_{k=1}^{n} \left(\frac{a_k}{b_k} - a_{k-1} \right) \prod_{j=k}^{n} b_j = \sum_{k=1}^{n-1} \left[a_k \prod_{j=k+1}^{n} b_j - a_{k-1} \prod_{j=k}^{n} b_j \right] + \left(\frac{a_n}{b_n} - a_{n-1} \right) b_n$$

$$= \quad a_{n-1} b_n - a_0 \prod_{j=1}^{n} b_j + a_n - a_{n-1} b_n = a_n.$$

We also use the identities:

(1) $F_{2n} = F_n L_n$ (see [1](13));
(2) $L_n = F_{n+1} + F_{n-1}$ (see [1](6));
(3) $G_{n+1}F_m + G_n F_{m-1} = G_{n+m}$ (see [1](8)).

(i) If $a_n = F_{2^{n+1}-2}$ and $b_n = F_{2^n-1}$, then $a_k/b_k - a_{k-1}$ equals

$$\frac{F_{2^{k+1}-2}}{F_{2^k-1}} - F_{2^k-2} = L_{2^k-1} - F_{2^k-2} = F_{2^k} \quad \text{by (1) and (2).}$$

Using the lemma, we obtain

$$\sum_{k=1}^{n} F_{2^k} \prod_{j=k}^{n} F_{2^j-1} = F_{2^{n+1}-2}.$$

(ii) If $a_n = F_{2^{n+1}-2}$ and $b_n = L_{2^n-1}$, then $a_k/b_k - a_{k-1}$ equals

$$\frac{F_{2^{k+1}-2}}{L_{2^k-1}} - F_{2^k-2} = F_{2^k-1} - F_{2^k-2} = F_{2^k-3} \quad \text{by (1).}$$

Using the lemma, we obtain

$$\sum_{k=1}^{n} F_{2^k-3} \prod_{j=k}^{n} L_{2^j-1} = F_{2^{n+1}-2}.$$

(iii) If $a_n = (-1)^n F_{2^{n+1}-2}$ and $b_n = L_{2^n-1}$, then $a_k/b_k - a_{k-1}$ equals

$$\frac{(-1)^k F_{2^{k+1}-2}}{L_{2^k-1}} - (-1)^{k-1} F_{2^k-2} = (-1)^k F_{2^k-1} + (-1)^k F_{2^k-2} = (-1)^k F_{2^k} \quad \text{by (1).}$$

Using the lemma, we obtain

$$\sum_{k=1}^{n} (-1)^k F_{2^k} \prod_{j=k}^{n} L_{2^j-1} = (-1)^n F_{2^{n+1}-2}.$$

(iv) If $a_n = (-1)^n G_{n+1} F_{2^{n+1}-2}$ and $b_n = L_{2^n-1}$, then $a_k/b_k - a_{k-1}$ equals

$$\begin{aligned} \frac{(-1)^k G_{k+2} F_{2^{k+1}-2}}{L_{2^k-1}} - (-1)^k G_{k+1} F_{2^k-2} &= (-1)^k (G_{k+2} F_{2^k-1} + G_{k+1} F_{2^k-2}) \\ &= (-1)^k G_{2^k+k} \quad \text{by (3).} \end{aligned}$$

Using the lemma, we obtain

$$\sum_{k=1}^{n} (-1)^k G_{2^k+k} \prod_{j=k}^{n} L_{2^j-1} = (-1)^n G_{n+2} F_{2^{n+1}-2}.$$

Note by the proposer: Putting $a_n = F^2_{2^{n+1}-2}$ and $b_n = L^2_{2^n-1}$ in the lemma, we get

$$\sum_{k=1}^{n} F_{2^k} F_{2^k-3} \prod_{j=k}^{n} L^2_{2^j-1} = F^2_{2^{n+1}-2}.$$

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Also solved by Dmitry Fleischman and Raphael Schumacher.

Some telescopic series

H-818 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 56, No. 1, February 2018)

Determine

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+4}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2} F_{n+3} F_{n+4}}.$$

Solution by the proposer

Let

$$S = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+4}} \quad \text{and} \quad T = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2} F_{n+3} F_{n+4}}.$$

We have

$$\begin{aligned} S + T &= \sum_{n=1}^{\infty} \frac{F_{n+3} + F_{n+1}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \\ &= \sum_{n=1}^{\infty} \frac{F_{n+4} - F_{n+1}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} - \frac{1}{F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \right) \\ &= \frac{1}{F_1 F_2 F_3 F_4} = \frac{1}{6}, \end{aligned}$$

and

$$\begin{aligned} S - T &= \sum_{n=1}^{\infty} \frac{F_{n+3} - F_{n+1}}{F_n F_{n+1} F_{n+2} Fn + 3 F_{n+4}} \\ &= \sum_{n=1}^{\infty} \frac{F_{n+2}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} \\ &= \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+3} F_{n+4}} \\ &= \sum_{n=3}^{\infty} \frac{1}{F_{n-2} F_{n-1} F_{n+1} F_{n+2}} \\ &= \sum_{n=3}^{\infty} \frac{1}{F_n^4 - 1} \qquad \text{(by Gelin-Cesàro identity)} \\ &= \frac{35 - 15\sqrt{5}}{18} \quad \text{(from Advanced Problem } \textbf{H-783}\text{(iii))}. \end{aligned}$$

Therefore, we obtain

$$S = \frac{38 - 15\sqrt{5}}{36} \quad \text{and} \quad T = \frac{-32 + 15\sqrt{5}}{36}.$$

Also solved by Dmitry Fleichman, Raphael Schumacher, and Albert Stadler.

Evaluating a definite integral

H-819 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania (Vol. 56, No. 1, February 2018)

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and odd function and $g : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ be a continuous function such that $g(1/x) = -g(x)$ for all $x \in \mathbb{R}_+^*$. Compute

$$\int_{-\beta}^{\alpha} \frac{dx}{(1+x^2)(1+e^{(f\circ g)(x)})},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Solution by Ravi Prakash, New Delhi, India

Note that

$$\beta = \frac{1-\sqrt{5}}{2} = -\frac{2}{\sqrt{5}+1} = -\frac{1}{\alpha} \qquad \text{so} \qquad \beta = -\frac{1}{\alpha}.$$

Also,

$$(f\circ g)\Big(\frac{1}{t}\Big) = f\Big(g\Big(\frac{1}{t}\Big)\Big) = f(-g(t)) = -f(g(t)) = -(f\circ g)(t).$$

Let

$$I = \int_{-\beta}^{\alpha} \frac{dx}{(1+x^2)[1+e^{(f\circ g)(x)}]}. \tag{1}$$

Make the substitution

$$x = \frac{1}{t}, \qquad dx = -\frac{1}{t^2}dt$$

to get

$$I = \int_{\alpha}^{\frac{1}{\alpha}} \frac{(-1/t^2)dt}{(1+1/t^2)[1+e^{-(f\circ g)(t)}]} = \int_{\frac{1}{\alpha}}^{\alpha} \frac{e^{(f\circ g)(x)}dx}{(1+x^2)[1+e^{(f\circ)(x)}]}. \tag{2}$$

Adding (1) and (2), we get

$$\begin{aligned} 2I &= \int_{\frac{1}{\alpha}}^{\alpha} \frac{1+e^{(f\circ g)(x)}}{(1+x^2)[1+e^{(f\circ g)(x)}]}dx \\ &= \int_{\frac{1}{\alpha}}^{\alpha} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_{x=\frac{1}{\alpha}}^{x=\alpha} = \tan^{-1}(\alpha) - \tan^{-1}\Big(\frac{1}{\alpha}\Big) \\ &= \tan^{-1}\left[\frac{\alpha - (\frac{1}{\alpha})}{1+\alpha(\frac{1}{\alpha})}\right] = \tan^{-1}\Big(\frac{1}{2}\Big). \end{aligned}$$

Therefore,

$$I = \frac{1}{2}\tan^{-1}\Big(\frac{1}{2}\Big).$$

Also solved by Dmitry Fleischman, Robert Frontczak, Ángel Plaza, Santiago Alzate Suarez and Kevin Darío López Rodríguez (jointly), Raphael Schumacher, Albert Stadler, Nicuşor Zloata, and the proposers.

Evaluating a Fibonacci limit

H-820 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania

If $a, b, c \in \mathbb{R}_+$, compute

$$\lim_{n\to\infty} \frac{\left(\sqrt[n+1]{(2n+1)!!F_{n+1}^b}\right)^{a+1} - \left(\sqrt[n]{(2n-1)!!F_n^b}\right)^{a+1}}{\left(\sqrt[n]{n!L_n^c}\right)^a}.$$

Solution by the proposers

We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!F-n^b}}{n} &= \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!F_n^b}{n^n}} \\
&= \lim_{n\to\infty} \frac{(2n+1)!!F_{n+1}^b}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!F_n^b} \quad \text{(by Cauchy-D'Alembert)} \\
&= \lim_{n\to\infty} \left(\frac{2n+1}{n+1}\right)\left(\frac{F_{n+1}}{F_n}\right)^b \left(\frac{n}{n+1}\right)^n = \frac{2\alpha^b}{e},
\end{aligned}$$

where as usual, $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Analogously, we obtain that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!L_n^c}}{n} = \frac{\alpha^c}{e}.$$

We denote

$$u_n = \frac{\sqrt[n+1]{(2n+1)!!F_{n+1}^b}}{\sqrt[n]{(2n-1)!!F_n^b}}.$$

We have

$$\begin{aligned}
\lim_{n\to\infty} u_n &= \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{(2n+1)!!F_{n+1}^b}}{n+1}\right)\left(\frac{n}{\sqrt[n]{(2n-1)!!F_n^b}}\right) \cdot \left(\frac{n+1}{n}\right) \\
&= \frac{2\alpha^b}{e} \cdot \frac{e}{2\alpha^b} \cdot 1,
\end{aligned}$$

so $\lim_{n\to\infty} u_n = 1$. In particular,

$$\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1.$$

Next,

$$\begin{aligned}
\lim_{n\to\infty} u_n^n &= \lim_{n\to\infty} \frac{(2n+1)!!F_{n+1}^b}{(2n-1)!!F_n^b} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!F_{n+1}^b}} \\
&= \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)^b \left(\frac{2n+1}{n+1}\right) \frac{n+1}{\sqrt[n+1]{(2n+1)!!F_{n+1}^b}} \\
&= \alpha^b \cdot 2 \cdot \frac{e}{2\alpha^b} = e.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\left(\sqrt[n+1]{(2n+1)!!F_{n+1}^b}\right)^{a+1} - \left(\sqrt[n]{(2n-1)!!F_n^b}\right)^{a+1}}{\left(\sqrt[n]{n!L_n^c}\right)^a} \\
= &\lim_{n\to\infty} \frac{(\sqrt[n]{(2n-1)!!F_n^b})^{a+1}}{(\sqrt[n]{n!L_n^c})^a}(u_n^{a+1}-1) \\
= &\lim_{n\to\infty} \frac{\left(\sqrt[n]{(2n+1)!!F_{n+1}^b}\right)^{a+1}}{(\sqrt[n]{n!L_n^c})^a} \cdot \frac{u_n^{a+1}-1}{\ln u_n^{a+1}} \cdot \ln u_n^{a+1} \\
= &\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{(2n+1)!!F_{n+1}^b}}{n+1}\right)^{a+1} \left(\frac{n}{\sqrt[n]{n!L_n^c}}\right)^a \left(\frac{n+1}{n}\right)^{a+1} \frac{u_n^{a+1}-1}{\ln u_n^{a+1}} \cdot \ln u_n^{n(a+1)} \\
= &\ \frac{2^{a+1}\alpha^{b(a+1)}}{e^{a+1}} \cdot \frac{e^a}{\alpha^{ca}} \cdot 1 \cdot 1 \cdot \ln(\lim_{n\to\infty}(u_n^n)^{a+1}) \\
= &\ \frac{2^{a+1}}{e}\alpha^{(b-c)a+b} \cdot (a+1)\ln(\lim_{n\to\infty} u_n^n) \\
= &\ \frac{(a+1)2^{a+1}\alpha^{a(b-c)+b}}{e}.
\end{aligned}$$

Also solved by Kenny B. Davenport, Dmitry Fleischman, and Raphael Schumacher.

Late Acknowledgement: Kenny B. Davenport solved Advanced Problem **H-811**. The Editor apologies for the oversight.