ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

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PROBLEMS PROPOSED IN THIS ISSUE

H-829 Proposed by Ángel Plaza and Francisco Perdomo, Gran Canaria, Spain

For any positive integer k, let $\{F_{k,n}\}_{n\geq 0}$ be the sequence defined by $F_{k,0}=0$, $F_{k,1}=1$, and $F_{k,n+1}=F_{k,n}+F_{k,n-1}$ for $n\geq 1$. Find the limit

$$\lim_{k\to\infty}\frac{k+\sqrt{k^2+4}}{2}\sum_{n=1}^{\infty}\arctan\left(\frac{kF_{k,n+1}^2}{1+F_{k,n}F_{k,n+1}^2F_{k,n+2}}\right).$$

H-830 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For an integer $n\geq 1$, prove that

$$12\sum_{k=1}^{n}(F_kF_{k+1}F_{k+2})^2\equiv 0 \pmod{F_nF_{n+1}F_{n+2}F_{n+3}}.$$

H-831 Proposed by Predrag Terzić, Podgorica, Montenegro

Let $P_j(x)=2^{-j}((x-\sqrt{x^2-4})^j+(x+\sqrt{x^2-4})^j)$, where j and x are nonnegative integers. Let $N=k2^m+1$ with k odd, $k<2^m$ and $m>2$. Let $S_0=P_k(F_n)$ and $S_i=S_{i-1}^2-2$ for $i\geq 1$. Prove the following statement: If there exists F_n for which $S_{m-2}\equiv 0 \pmod{N}$, then N is prime.

H-832 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For positive integers n and r, find a closed form expressions for

(i) $\sum_{k=1}^{n}F_{rk}^3L_{rk}$;
(ii) $\sum_{k=1}^{n}F_{2F_k}^3F_{2L_k}$.

SOLUTIONS

Diophantine equations with powers of the golden section

H-796 Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Florian Luca, Johannesburg, South Africa (Vol. 54, No. 3, August 2016)

Find all solutions (x, y) in positive integers of the equation

$$\tan^{-1}\alpha^x - \tan^{-1}\alpha^y = \tan^{-1}x - \tan^{-1}y,$$

where α is the golden section.

Solution by the proposers

We show that $(x, y) = (7, 5)$ is the only solution. Applying tangent in both sides of the equation, the left side of it becomes

$$\tan\left(\tan^{-1}\alpha^x - \tan^{-1}\alpha^y\right) = \frac{\alpha^x - \alpha^y}{1+\alpha^{x+y}}, \tag{1}$$

whereas the right side of it becomes

$$\tan\left(\tan^{-1}x - \tan^{-1}y\right) = \frac{x-y}{1+xy}. \tag{2}$$

Since (2) is rational, (1) should be invariant by the action of the only nontrivial Galois automorphism of $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, which sends α to $\beta = -\alpha^{-1}$. Applying this to (1), we get that (1) is

$$\frac{\beta^x - \beta^y}{1+\beta^{x+y}} = \frac{(-1)^x\alpha^{-x} - (-1)^y\alpha^{-y}}{1+(-1)^{x+y}\alpha^{-x-y}} = \frac{(-1)^x\alpha^{y} - (-1)^y\alpha^{x}}{\alpha^{x+y}+(-1)^{x+y}}. \tag{3}$$

Assume first that $x+y$ is odd. Then, $(-1)^x = (-1)^{y+1}$ and from (1) and (3), we get

$$\frac{\alpha^x - \alpha^y}{\alpha^{x+y}+1} = (-1)^x\frac{\alpha^y + \alpha^x}{\alpha^{x+y}-1}.$$

Since $x > y$, the left side above is positive. Thus, x is even and we get

$$\frac{\alpha^x - \alpha^y}{\alpha^{x+y}+1} = \frac{\alpha^x + \alpha^y}{\alpha^{x+y}-1} \quad \text{or, equivalently} \quad \frac{\alpha^{x+y}-1}{\alpha^{x+y}+1} = \frac{\alpha^x + \alpha^y}{\alpha^x - \alpha^y},$$

which is false since in the last equality of fractions, the left side is < 1 whereas the right side is > 1. So, $x+y$ is even, therefore $(-1)^x = (-1)^y$. So, we get from (1) and (3) that

$$\frac{\alpha^x - \alpha^y}{\alpha^{x+y}+1} = (-1)^{x+1}\frac{\alpha^x - \alpha^y}{\alpha^{x+y}+1},$$

so $x+1$ is even. Hence, x is odd and y is odd. Now, the given equation is

$$\frac{x-y}{1+xy} = \frac{\alpha^x - \alpha^y}{\alpha^{x+y}+1} = \frac{\alpha^{(x-y)/2} - \alpha^{(y-x)/2}}{\alpha^{(x+y)/2} + \alpha^{-(x+y)/2}} = \begin{cases} \frac{L_{(x-y)/2}}{L_{(x+y)/2}} & \text{if} \quad x+y \equiv 0 \pmod 4; \\ \frac{F_{(x-y)/2}}{F_{(x+y)/2}} & \text{if} \quad x+y \equiv 2 \pmod 4. \end{cases}$$

Recall that $\gcd(F_a, F_b) = F_d$, where $d = \gcd(a, b)$. Further, $\gcd(L_a, L_b)$ equals L_d if and only if a/d and b/d are both odd. In the contrary case, $\gcd(L_a, L_b) \in \{1, 2\}$. So, if $x+y \equiv 0 \pmod 4$, then $(x-y)/2$ is odd and $(x+y)/2$ is even, therefore $\gcd(L_{(x-y)/2}, L_{(x+y)/2}) \in \{1, 2\}$. Thus, the denominator of the reduced fraction $L_{(x-y)/2}/L_{(x+y)/2}$ is at least $L_{(x+y)/2}/2$. In case $x+y \equiv 2 \pmod 4$, we have that $(x-y)/2$ is even and $(x+y)/2$ is odd, therefore $\gcd(F_{(x-y)/2}, F_{(x+y)/2}) \le F_{(x-y)/4}$, so the denominator of the reduced fraction $F_{(x-y)/2}/F_{(x+y)/2}$ is at least $F_{(x+y)/2}/F_{(x-y)/4}$.

Next, we recall that

$$\alpha^{n-2} \le F_n \le \alpha^{n-1} \quad \text{and} \quad \alpha^{n-1} \le L_n \le \alpha^{n+1}$$

hold for all $n \ge 1$. Thus, in case $x + y \equiv 0 \pmod 4$, the denominator of $L_{(x-y)/2}/L_{(x+y)/2}$ is at least as large as

$$\frac{L_{(x+y)/2}}{2} \ge \frac{\alpha^{(x+y)/2-1}}{2} > \alpha^{x/2-3},$$

whereas in case $x + y \equiv 2 \pmod 4$, the denominator of $F_{(x-y)/2}/F_{(x+y)/2}$ is at least as large as

$$\alpha^{(x+y)/2-2-((x-y)/4-1)} = \alpha^{x/4+3y/4-1} > \alpha^{x/4-1}.$$

At any rate, this denominator is also the denominator of $(x - y)/(1 + xy)$ and is therefore $\le 1 + xy < x^2$. We thus get that

$$x^2 > \min\left\{\alpha^{x/2-3}, \alpha^{x/4-1}\right\},$$

which leads to $x \le 75$. So, all solutions have $1 \le y < x \le 75$ and we finish with a computer search.

For other equations with the inverse tangent of powers of the golden section, see [1].

[1] F. Luca and P. Stănică, *On Machin's formula with powers of the golden section*, Int. J. Number Theory, **5** (2009), 973–979.

Partially solved by Dmitry Fleischman.

An identity with Fibonomial coefficients

H-797 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 54, No. 4, November 2016)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For positive integers a, b, c, and $d = a+b+c-1$, prove that

$$\sum_{k=0}^{a} F_{2k}\binom{2a}{a+k}_F\binom{2b}{b+k}_F\binom{2c}{c+k}_F\binom{2d}{d+k}_F^{-1} = \frac{F_aF_bF_cF_{d+1}}{F_{a+b}F_{b+c}F_{c+a}}\binom{2a}{a}_F\binom{2b}{b}_F\binom{2c}{c}_F\binom{2d}{d}_F^{-1}.$$

Solution by the proposer

Let

$$\begin{aligned}\Delta(k) &= \binom{2a-1}{a+k-1}_F\binom{2b-1}{b+k-1}_F\binom{2c-1}{c+k-1}_F\binom{2d+1}{d+k}_F^{-1}\\ &- \binom{2a-1}{a+k}_F\binom{2b-1}{b+k}_F\binom{2c-1}{c+k}_F\binom{2d+1}{d+k+1}_F^{-1}.\end{aligned}$$

In [1], Melham showed that

$$F_{k+a+b+c}F_{k-a}F_{k-b}F_{k-c} - F_{k-a-b-c}F_{k+a}F_{k+b}F_{k+c} = (-1)^{k+a+b+c}F_{a+b}F_{b+c}F_{c+a}F_{2k}.$$

That is,

$$F_{a+k}F_{b+k}F_{c+k}F_{d+1-k} - F_{a-k}F_{b-k}F_{c-k}F_{d+1+k} = F_{a+b}F_{b+c}F_{c+a}F_{2k}. \tag{4}$$

We have

$$\begin{aligned}\Delta(k) &= \frac{F_{a+k}}{F_{2a}}\binom{2a}{a+k}_F \frac{F_{b+k}}{F_{2b}}\binom{2b}{b+k}_F \frac{F_{c+k}}{F_{2c}}\binom{2c}{c+k}_F \frac{F_{d-k+1}}{F_{2d+1}}\binom{2d}{d+k}_F^{-1} \\ &- \frac{F_{a-k}}{F_{2a}}\binom{2a}{a+k}_F \frac{F_{b-k}}{F_{2b}}\binom{2b}{b+k}_F \frac{F_{c-k}}{F_{2c}}\binom{2c}{c+k}_F \frac{F_{d+k+1}}{F_{2d+1}}\binom{2d}{d+k}_F^{-1} \\ &= \frac{F_{a+b}F_{b+c}F_{c+a}F_{2k}}{F_{2a}F_{2b}F_{2c}F_{2d+1}}\binom{2a}{a+k}_F\binom{2b}{b+k}_F\binom{2c}{c+k}_F\binom{2d}{d+k}_F^{-1}\end{aligned}$$

by (4). Therefore, we have

$$\begin{aligned}&\sum_{k=0}^{a} F_{2k}\binom{2a}{a+k}_F\binom{2b}{b+k}_F\binom{2c}{c+k}_F\binom{2d}{d+k}_F^{-1} = \frac{F_{2a}F_{2b}F_{2c}F_{2d+1}}{F_{a+b}F_{b+c}F_{c+a}}\sum_{k=0}^{a}\Delta(k) \\ &= \frac{F_{2a}F_{2b}F_{2c}F_{2d+1}}{F_{a+b}F_{b+c}F_{c+a}}\left(\binom{2a-1}{a-1}_F\binom{2b-1}{b-1}_F\binom{2c-1}{c-1}_F\binom{2d+1}{d}_F^{-1}\right. \\ &- \left.\binom{2a-1}{2a}_F\binom{2b-1}{b+a}_F\binom{2c-1}{c+a}_F\binom{2d+1}{d+a+1}_F^{-1}\right) \\ &= \frac{F_{2a}F_{2b}F_{2c}F_{2d+1}}{F_{a+b}F_{b+c}F_{c+a}}\binom{2a-1}{a-1}_F\binom{2b-1}{b-1}_F\binom{2c-1}{c-1}_F\binom{2d+1}{d}_F^{-1} \\ &= \frac{F_{2a}F_{2b}F_{2c}F_{2d+1}}{F_{a+b}F_{b+c}F_{c+a}} \times \frac{F_a}{F_{2a}}\binom{2a}{a}_F\frac{F_b}{F_{2b}}\binom{2b}{b}_F\frac{F_c}{F_{2c}}\binom{2c}{c}_F\frac{F_{d+1}}{F_{2d+1}}\binom{2d}{d}_F^{-1} \\ &= \frac{F_aF_bF_cF_{a+b+c}}{F_{a+b}F_{b+c}F_{c+a}}\binom{2a}{a}_F\binom{2b}{b}_F\binom{2c}{c}_F\binom{2d}{d}_F^{-1}.\end{aligned}$$

The proposer also noticed that in the same manner

$$\begin{aligned}&\sum_{k=0}^{a} F_{2k}\binom{a+b}{a+k}_F\binom{b+c}{b+k}_F\binom{c+a}{c+k}_F\binom{2d}{d+k}_F^{-1} \\ &= \frac{F_aF_bF_cF_{a+b+c}}{F_{a+b}F_{b+c}F_{c+a}}\binom{a+b}{a}_F\binom{b+c}{b}_F\binom{c+a}{c}_F\binom{2d}{d}_F^{-1}.\end{aligned}$$

[1] R. S. Melham, *On product difference Fibonacci identities*, INTEGERS, **11** (2010), #A10.

Also partially solved by Dmitry Fleischman.

An inequality with Fibonacci numbers and trigonometric functions

H-798 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania (Vol. 54, No. 4, November 2016)

If $t \in (0, \pi/2)$ and $m \geq 0$, prove that

$$\frac{\sin^{m+2} t}{(F_n \sin t + F_{n+1}\cos t)^m} + \frac{\cos^{m+2} t}{(F_n \cos t + F_{n+1}\sin t)^m} \geq \frac{1}{F_{n+2}^m}$$

and

$$\frac{1}{(L_n + L_{n+1}\tan t)^m} + \frac{\tan^{m+2} t}{(L_n \tan t + L_{n+1})^m} \geq \frac{1}{L_{n+2}^m \cos^2 t}$$

hold for all $n \geq 1$.

Solution by Soumitra Mondal

We have

$$\begin{aligned}
&\frac{\sin^{m+2} t}{(F_n \sin t + F_{n+1}\cos t)^m} + \frac{\cos^{m+2} t}{(F_n \cos t + F_{n+1}\sin t)^m}\\
= \;& \frac{(\sin^2 t)^{m+1}}{(F_n \sin^2 t + F_{n+1}\sin t\cos t)^m} + \frac{(\cos^2 t)^{m+1}}{(F_n \cos^2 t + F_{n+1}\sin t\cos t)^m}\\
\geq \;& \frac{(\sin^2 t + \cos^2 t)^{m+1}}{(F_n + 2F_{n+1}\sin t\cos t)^m} \qquad \text{(by Radon's inequality)}\\
\geq \;& \frac{1}{(F_n + F_{n+1}(\sin^2 t + \cos^2 t))^m} = \frac{1}{(F_n + F_{n+1})^m} = \frac{1}{F_{n+2}^m}.
\end{aligned}$$

Again

$$\begin{aligned}
&\frac{1}{(L_n + L_{n+1}\tan t)^m} + \frac{\tan^{m+2} t}{(L_n \tan t + L_{n+1})^m}\\
= \;& \frac{1}{(L_n + L_{n+1}\tan t)^m} + \frac{(\tan^2 t)^{m+1}}{(L_n \tan^2 t + L_{n+1}\tan t)^m}\\
\geq \;& \frac{(1+\tan^2 t)^{m+1}}{(L_n \sec^2 t + 2L_{n+1}\tan t)^m} \qquad \text{(by Radon's inequality)}\\
\geq \;& \frac{\sec^{2m+2} t}{(L_n \sec^2 t + L_{n+1}(1+\tan^2 t))^m} = \frac{1}{(L_n + L_{n+1})^m \cos^2 t} = \frac{1}{L_{n+2}^m \cos^2 t}.
\end{aligned}$$

Also solved by Dmitry Fleischman and the proposers.

An inequality with Fibonacci numbers

H-799 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania (Vol. 54, No. 4, November 2016)

Prove that

$$\frac{F_n}{F_{n+1}(F_{n+1}^2 + 4F_nF_{n+1} + 3F_n^2)} + \frac{F_{n+1}}{F_n(3F_{n+1}^2 + 4F_nF_{n+1} + F_n^2)} \geq \frac{4F_nF_{n+1}}{F_{n+2}^4}$$

and that the same inequality with all F's replaced by L's holds for all $n \geq 1$.

Solution by Brian Bradie

Note that

$$F_{n+1}^2 + 4F_nF_{n+1} + 3F_n^2 = (F_{n+1}+F_n)(F_{n+1}+3F_n) = F_{n+2}(F_{n+1}+3F_n),$$

and

$$3F_{n+1}^2 + 4F_nF_{n+1} + F_n^2 = (3F_{n+1}+F_n)(F_{n+1}+F_n) = F_{n+2}(3F_{n+1}+F_n).$$

Then, the desired inequality is equivalent to

$$F_{n+2}^3(F_n^2(3F_{n+1}+F_n) + F_{n+1}^2(3F_n+F_{n+1})) \geq 4F_n^2F_{n+1}^2(3F_n+F_{n+1})(3F_{n+1}+F_n).$$

This in turn is equivalent to

$$(F_n - F_{n+1})^2(F_n^2 + 4F_nF_{n+1} + F_{n+1}^2)^2 \geq 0,$$

which is clearly true. Moreover, inequality holds if and only if $n = 1$.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Wei Kai-Lai and John Risher (jointly), Soumitra Mondal, Ángel Plaza, Hideyuki Ohtsuka, and the proposers.

A sum involving multinomial coefficients

H-800 Proposed by Mehtaab Sawhney, Commack, NY (Vol. 54, No. 4, November 2016)

Let

$$S_k = \sum_{\substack{n_1+2n_2+\cdots+kn_k=k \\ n_1,n_2,\ldots,n_k\in\mathbb{Z}_{\geq 0}}} (-1)^{n_1+n_2+\cdots+n_k} \binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\ldots,n_k} \prod_{j=1}^{k} (j+1)^{n_j}.$$

Compute S_1, S_2 and show that $S_k = 0$ for all $k \geq 3$.

Solution by Eduardo H. M. Brietzke

We generalize the argument presented in [1], page 38, introducing parameters.

Claim 1: If $x_1, x_2, \ldots$ is any infinite set of parameters then

$$\sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_1+k_2+\cdots+k_n=r}} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} x_1^{k_1}\cdots x_n^{k_n} = [q^n]\left(\sum_{j\geq 1} x_j q^j\right)^r. \tag{5}$$

In the above, for a formal series $\sum_{n\geq 0} a_n q^n$, the notation $[q^n]\sum_{n\geq 0} a_n q^n$ stands for the coefficient of q^n (equal to a_n).

Indeed, let $\psi(q) = e^{tq} = \sum_{n=0}^{\infty} \frac{t^n}{n!} q^n$ and consider the infinite product of formal power series $P := \prod_{j\geq 1} \psi(x_j q^j)$. Then,

$$\begin{aligned} P &= \prod_{j\geq 1}\sum_{k\geq 0} \frac{t^k x_j^k q^{jk}}{k!} = \sum_{n\geq 0} q^n \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{t^{k_1+\cdots+k_n} x_1^{k_1}\cdots x_n^{k_n}}{k_1!\cdots k_n!} \\ &= \sum_{n\geq 0} q^n \sum_{r\geq 0} t^r \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_1+k_2+\cdots+k_n=r}} \frac{x_1^{k_1}\cdots x_n^{k_n}}{k_1!\cdots k_n!}. \end{aligned} \tag{6}$$

Also,

$$P = e^{t\sum_{j\geq 1} x_j q^j} = \sum_{r\geq 0} \frac{t^r}{r!}\left(\sum_{j\geq 1} x_j q^j\right)^r \tag{7}$$

Comparing the coefficient of $q^n t^r$ in (6) and (7), we obtain (5).

Applying summation on r from 1 to infinity to both sides of (5), it follows that

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} x_1^{k_1}\cdots x_n^{k_n} = [q^n]\left(\left(1-\sum_{j\geq 1} x_j q^j\right)^{-1} - 1\right). \tag{8}$$

Now, applying (8) with $x_j = -(j+1)$, we get

$$\sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+k_2+\cdots+k_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} 2^{k_1} 3^{k_2}\cdots n^{k_n} = [q^n]\left(\frac{1}{1+2q+3q^2+\cdots} - 1\right),$$

or,

$$S_n = [q^n]((1-q)^2 - 1),$$

from which it follows that $S_1 = -2$, $S_2 = 1$, and $S_n = 0$ for $n \geq 3$.

Remark. Other choices of values for x_j in (8) might yield interesting results as well. For example, for $x_j = -(j+1)^2$ we get

$$\sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+k_2+\cdots+k_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} 2^{2k_1}3^{2k_2}\cdots(n+1)^{2k_n} = [q^n]\left(\frac{(1-q)^3}{1+q} - 1\right)$$

$$= \begin{cases} 0, & \text{if } n = 0 \\ -4, & \text{if } n = 1 \\ 7, & \text{if } n = 2 \\ (-1)^n 8, & \text{if } n \geq 3 \end{cases}$$

[1] N. J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs 27, AMS 1988.

Also solved by Dmitry Fleishman and the proposer.

Errata: In the statement of **H-827** the last factor inside the inner limit should be "$n^{F_{m-1}/F_{m+1}}$" instead of "n^{F_{m-1}/F_m}".