ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-846 Proposed by Robert Frontczak, Stuttgart, Germany

Let $T_n = n(n+1)/2$ be the nth triangular number. Further, let $(G_n)_{n\geq 0}$ be the Tribonacci sequence given by $G_0 = 0$, $G_1 = G_2 = 1$, and for $n \geq 3$,

$$G_n = G_{n-1} + G_{n-2} + G_{n-3}.$$

Prove that

$$\sum_{k=0}^{n} T_k G_{n-k} = \frac{1}{2}(G_{n+4} - T_{n+4}) + (n+3).$$

Extend this result to a generalized Tribonacci sequence with $G_0 = a$, $G_1 = b$, and $G_2 = c$.

H-847 Proposed by Robert Frontczak, Stuttgart, Germany

Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers. Further, let $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ be the Balancing and Lucas-Balancing numbers, respectively; i.e., for all $n \geq 1$, we have

$$B_{n+1} = 6B_n - B_{n-1} \qquad \text{and} \qquad C_{n+1} = 6C_n - C_{n-1},$$

with $B_0 = 0$, $B_1 = 1$, $C_0 = 1$, $C_1 = 3$. Show that for all $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^{k+1} B_{n-k} F_{2k}^3 = \frac{(-1)^n}{24} F_{2(n-1)} F_{2n} F_{2(n+1)},$$

and

$$\sum_{k=0}^{n} (-1)^{k} C_{n-k} F_{2k}^3 = \frac{(-1)^n}{8} \left(F_{2(n-1)} F_{2n} F_{2(n+1)} + \frac{F_{2n} F_{2(n+1)} F_{2(n+2)}}{3} \right).$$

H-848 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $p \geq 2$ be a real number. The sequence $(U_n)_{n\geq 0}$ is defined by

$$U_0 = 0, \quad U_1 = 1, \quad \text{and} \quad U_{n+2} = pU_{n+1} - U_n \quad \text{for all} \quad n \geq 0.$$

Prove that

$$\lim_{n\to\infty}\sqrt{1+U_1\sqrt{1+U_2\sqrt{1+U_3\sqrt{\cdots+U_{n-1}\sqrt{1+U_n}}}}}=p.$$

H-849 **Proposed by Hideyuki Ohtsuka, Saitama, Japan**

For nonnegative integers m and n, find a closed form formula for

$$\sum_{\substack{i+j+k=n\\ i,j,k\geq 0}}(-1)^j L_{i-j}\binom{m}{k}\binom{n}{i,j,k}.$$

SOLUTIONS

Traces of powers of the Pascal triangle matrix

H-609 **Proposed by Mario Catalani, University of Torino, Italy (Vol. 42, No. 1, February 2004)**

Let $\mathbf{R}_n$ be the right-justified Pascal-triangle matrix, that is the $n\times n$ matrix with the (i,j) element r_{ij}, $1\leq i,j\leq n$, given by

$$r_{ij}=\binom{i-1}{n-j}.$$

a. Find $\operatorname{tr}(\mathbf{R}_n^m)$, where $\operatorname{tr}(\cdot)$ is the trace operator and m is a positive integer.
b. Find $|\mathbf{R}_n+\mathbf{R}_n^{-1}|$, where $|\cdot|$ is the determinant operator.

Solution by the Ilker Akkus and Gonca Kizilaslan

A matrix $\mathbf{A}_{m+1}$ of order $m+1$ is defined in [1] as

$$\mathbf{A}_{m+1}=\left[\binom{r}{m-s}\right]\qquad(r,s=0,1,\ldots,m). \tag{2}$$

We have seen that the matrix $\mathbf{R}_n$ defined in the text and the matrix $\mathbf{A}_n$ defined in (2) are the same matrix.

a. This part is solved in [1], that is

$$tr(\mathbf{R}_n^m)=\frac{F_{nm}}{F_m},$$

where F_n is the nth Fibonacci number.

b. We will find the product of the characteristic values of the matrix $\mathbf{R}_n+\mathbf{R}_n^{-1}$ to obtain $\left|\mathbf{R}_n+\mathbf{R}_n^{-1}\right|$. For this reason, we will use the characteristic values of the matrix $\mathbf{A}_{m+1}$ that are obtained by Carlitz [1]. Because $\mathbf{R}_n$ equals $\mathbf{A}_n$, we obtain the desired result.

In [1], the characteristic values of $\mathbf{A}_{m+1}$ are given as

$$\alpha^m,\ \alpha^{m-1}\beta,\ \ldots,\ \alpha\beta^{m-1},\ \beta^m,$$

where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$. Thus, the characteristic values of $\mathbf{A}_{m+1}+\mathbf{A}_{m+1}^{-1}$ are

$$\alpha^m+\frac{1}{\alpha^m},\ \alpha^{m-1}\beta+\frac{1}{\alpha^{m-1}\beta},\ \ldots,\ \alpha\beta^{m-1}+\frac{1}{\alpha\beta^{m-1}},\ \beta^m+\frac{1}{\beta^m}.$$

We will obtain these values in a more explicit form according to the parity of m. Let m be even, that is $m = 2k$. Then, the characteristic values will be:

$$\begin{aligned}
\alpha^{2k} + \frac{1}{\alpha^{2k}} &= L_{2k}, \\
\alpha^{2k-1}\beta + \frac{1}{\alpha^{2k-1}\beta} &= -L_{2k-2}, \\
\alpha^{2k-2}\beta^2 + \frac{1}{\alpha^{2k-2}\beta^2} &= L_{2k-4}, \\
&\vdots \\
\alpha^k\beta^k + \frac{1}{\alpha^k\beta^k} &= 2(-1)^k, \\
&\vdots \\
\alpha^2\beta^{2k-2} + \frac{1}{\alpha^2\beta^{2k-2}} &= L_{2k-4}, \\
\alpha\beta^{2k-1} + \frac{1}{\alpha\beta^{2k-1}} &= -L_{2k-2}, \\
\beta^{2k} + \frac{1}{\beta^{2k}} &= L_{2k},
\end{aligned}$$

where L_n is the nth Lucas number. Thus,

$$\left|\mathbf{A}_{m+1} + \mathbf{A}_{m+1}^{-1}\right| = 2(-1)^k \prod_{i=1}^{k} L_{2i}^2.$$

Let m be odd, that is $m = 2k - 1$. Then the characteristic values will be:

$$\begin{aligned}
\alpha^{2k-1} + \frac{1}{\alpha^{2k-1}} &= \sqrt{5}F_{2k-1}, \\
\alpha^{2k-2}\beta + \frac{1}{\alpha^{2k-2}\beta} &= -\sqrt{5}F_{2k-3}, \\
\alpha^{2k-3}\beta^2 + \frac{1}{\alpha^{2k-3}\beta^2} &= \sqrt{5}F_{2k-5}, \\
&\vdots \\
\alpha^k\beta^{k-1} + \frac{1}{\alpha^k\beta^{k-1}} &= \sqrt{5}(-1)^{k-1}, \\
\alpha^{k-1}\beta^k + \frac{1}{\alpha^{k-1}\beta^k} &= \sqrt{5}(-1)^k \\
&\vdots \\
\alpha^2\beta^{2k-3} + \frac{1}{\alpha^2\beta^{2k-3}} &= -\sqrt{5}F_{2k-5}, \\
\alpha\beta^{2k-2} + \frac{1}{\alpha\beta^{2k-2}} &= \sqrt{5}F_{2k-3}, \\
\beta^{2k-1} + \frac{1}{\beta^{2k-1}} &= -\sqrt{5}F_{2k-1}.
\end{aligned}$$

Thus, in this case we obtain

$$\left|\mathbf{A}_{m+1} + \mathbf{A}_{m+1}^{-1}\right| = 5^k \prod_{i=1}^{k} F_{2i-1}^2.$$

[1] L. Carlitz, *The characteristic polynomial of a certain matrix of binomial coefficients*, The Fibonacci Quarterly, **3.2** (1965), 81–89.

A Tribonacci identity

H-814 Proposed by Ray Melham, Sydney, Australia (Vol. 55, No. 4, November 2017)

Define the Tribonacci numbers, for all integers n, by $T_n = T_{n-1}+T_{n-2}+T_{n-3}$, with $T_{-1} = 0$, $T_0 = 0$, and $T_1 = 1$. If k and n are integers, prove that

$$\begin{aligned} -T_{2k}T_{n-2}^2 - T_{2k-2}T_{n-1}^2 - 2T_{2k-1}T_n^2 + 2(T_{2k} + T_{2k+1})T_{n+1}^2 \quad &+ \quad (T_{2k} + 2T_{2k+1})T_{n+2}^2 \\ &+ \quad T_{2k+2}T_{n+3}^2 = 2T_{2n+2k+4}. \end{aligned}$$

Solution by the Editor

By the Binet formula, we have that

$$T_k = c_1\alpha_1^k + c_2\alpha_2^k + c_3\alpha_3^k$$

for some numbers c_1, c_2, c_3, and all integers k. In particular,

$$\begin{aligned} -T_{2k}T_{n-2}^2 &= \alpha_1^{2k}(-c_1T_{n-2}^2) + \alpha_2^{2k}(-c_2T_{n-2}^2) + \alpha_3^{2k}(-c_3T_{n-2}^2); \\ -T_{2k-2}T_{n-1}^2 &= \alpha_1^{2k}(-c_1\alpha_1^{-2}T_{n-1}^2) + \alpha_2^{2k}(-c_2\alpha_2^{-2}T_{n-1}^2) + \alpha_3^{2k}(-c_3\alpha_3^{-2}T_{n-1}^2); \\ \vdots &= \vdots \\ T_{2k+2}T_{n+3}^2 &= \alpha_1^{2k}(c_1\alpha_1^2T_{n+3}^2) + \alpha_2^{2k}(c_2\alpha_2^2T_{n+3}^2) + \alpha_3^{2k}(c_3\alpha_3^2T_{n+3}^2); \\ 2T_{2n+2k+4} &= \alpha_1^{2k}(c_1\alpha_1^{2n+4}) + \alpha_2^{2k}(c_2\alpha_2^{2n+4}) + \alpha_3^{2k}(c_3\alpha_3^{2n+4}). \end{aligned}$$

This shows that

$$LS - RS = \alpha_1^{2k}C_1(n) + \alpha_2^{2k}C_2(n) + \alpha_3^{2k}C_3(n).$$

The coefficients $C_1(n)$, $C_2(n)$, $C_3(n)$ depend on n, but not on k. To show that $LS - RS$ is the constant 0, it suffices therefore to show that it is 0 for three consecutive values of k. Indeed, if the above expression is 0 for k, $k+1$, $k+2$, then $(C_1(n), C_2(n), C_3(n))^T$ is a solution of the homogeneous linear system whose matrix is

$$\begin{pmatrix} \alpha_1^{2k} & \alpha_2^{2k} & \alpha_3^{2k} \\ \alpha_1^{2k+2} & \alpha_2^{2k+2} & \alpha_3^{2k+2} \\ \alpha_1^{2k+4} & \alpha_2^{2k+4} & \alpha_3^{2k+4} \end{pmatrix}$$

and whose determinant is nonzero by known properties of Vandermonde determinants. Thus, $C_1(n) = C_2(n) = C_3(n) = 0$, which further shows that $LS - RS$ is 0 for all k (and this particular n). We choose $k \in \{-1, 0, 1\}$. Therefore, this reduces the problem to proving the following three identities for all n:

$$\begin{aligned} -T_{n-2}^2 + 2T_n^2 + 2T_{n+1}^2 + T_{n+2}^2 &= 2T_{2n+2}, \qquad k = -1; \\ -T_{n-1}^2 + 2T_{n+1}^2 + 2T_{n+2}^2 + T_{n+3}^2 &= 2T_{2n+4}, \qquad k = 0; \\ -T_{n-2}^2 - 2T_n^2 + 6T_{n+1}^2 + 5T_{n+2}^2 + 4T_{n+3}^2 &= 2T_{2n+6}, \qquad k = 1. \end{aligned} \tag{1}$$

The sequence with general term T_n^2 is linearly recurrent of order 6, and whose characteristic polynomial is

$$(X-\alpha_1^2)(X-\alpha_2^2)(X-\alpha_3^2)(X-\alpha_1\alpha_2)(X-\alpha_2\alpha_3)(X-\alpha_1\alpha_3) = X^6-2X^5-3X^4-6X^3+X^2+1. \tag{2}$$

Hence,

$$T_{n+6}^2 = 2T_{n+5}^2 + 3T_{n+4}^2 + 6T_{n+3}^2 - T_{n+2}^2 - T_n^2 \qquad \text{holds for all} \qquad n \in \mathbb{Z}. \tag{3}$$

Inspired by the first two relations (1), instead of the third relation (1), we write

$$\begin{aligned} & -T_n^2 + 2T_{n+2}^2 + 2T_{n+3}^2 + T_{n+4}^2 \\ = \; & -T_n^2 + 2T_{n+2}^2 + 2T_{n+3}^2 + (2T_{n+3}^2 + 3T_{n+2}^2 + 6T_{n+1}^2 - T_n^2 - T_{n-2}^2) \\ = \; & 4T_{n+3}^2 + 5T_{n+2}^2 + 6T_{n+1}^2 - 2T_n^2 - T_{n-2}^2, \end{aligned}$$

which we recognize as the left side of the third relation (1). In the above calculation, we used recurrence (3). So, the second and third relations (1) are equivalent to the first relation (1) with n replaced by $n+1$ and $n+2$, respectively. Thus, it suffices to show that the first relation (1) holds for all n.

By an argument similar to the one at the beginning of this proof, the sequence with general term

$$(-T_{n-2}^2 + 2T_n^2 + 2T_{n+1}^2 + T_{n+2}^2) - 2T_{2n+2}$$

is linearly recurrent of order 6 having the same characteristic polynomial as T_n^2, which is shown at (2). Thus, to show that it is the constant 0, it suffices to show that it is 0 for 6 consecutive values of n. So, we check that the first relation (1) holds for $n = 0, 1, 2, 3, 4, 5$, which finishes the proof.

Also solved by Dmitry Fleischman and Raphael Schumacher.

A congruence with powers of 2 and binomial coefficients

H-815 Proposed by Mehtaab Sawhney, Commack, NY (Vol. 55, No. 4, November 2017)

Let $p > 5$ be a prime congruent to 1 modulo 4. Prove that

$$\sum_{n=0}^{p-1} 2^n \binom{3n}{n} \equiv 0 \pmod{p}.$$

Solution by the proposer

Note that since

$$\left(\frac{-1}{p}\right) = 1$$

as $p \equiv 1 \pmod 4$, it follows that $i^2 = -1 \pmod p$ for some integer i. By Wilson's theorem, we can take $i = ((p-1)/2)!$. In particular,

$$x^2 + 4x + 8 = (x+2)^2 + 4 = (x + 2 + 2i)(x + 2 - 2i) \in \mathbb{Z}_p[x].$$

By partial fractions,

$$\frac{1}{(x+1)(x^2+4x+8)} = \sum_{n=0}^{\infty} \frac{x^n}{5\cdot 4^{n+2}}\Big[(-3-i)(-1-i)^n - (3-i)(-1+i)^n + (-1)^n 4^{n+2}\Big].$$

The above coefficients of x^n are well-defined modulo p for $p \equiv 1 \pmod 4$ and $p > 5$. Let $CT[f(x)]$ be the constant term of the series $f(x)$.

$$\begin{aligned}
&\sum_{n=0}^{p-1} 2^n \binom{3n}{n} \\
&= CT\left[\sum_{n=0}^{p-1} \frac{(x+2)^{3n}}{x^{2n}}\right] \\
&= CT\left[\frac{\left(\frac{(x+2)^{3p}}{x^{2p}}\right) - 1}{\left(\frac{(x+2)^{3}}{x^{2}}\right) - 1}\right] \\
&= CT\left[\frac{(x+2)^{3p} - x^{2p}}{x^{2p-2}((x+2)^3 - x^2)}\right] \\
&\equiv CT\left[\frac{(x^p+2)^{3} - x^{2p}}{x^{2p-2}((x+2)^3 - x^2)}\right] \pmod p \\
&\equiv CT\left[\frac{12x^p + 8}{x^{2p-2}(x+1)(x^2+4x+8)}\right] \pmod p \\
&\equiv CT\left[\frac{12x^p + 8}{x^{2p-2}} \sum_{n=0}^{\infty} \frac{x^n}{5\cdot 4^{n+2}}\Big[(-3-i)(-1-i)^n - (3-i)(-1+i)^n + (-1)^n 4^{n+2}\Big]\right] \pmod p \\
&\equiv 12\left[\frac{1}{5\cdot 4^p}\Big[(-3-i)(-1-i)^{p-2} - (3-i)(-1+i)^{p-2} + (-1)^{p-2}4^p\Big]\right] \\
&\quad + 8\left[\frac{1}{5\cdot 4^{2p}}\Big[(-3-i)(-1-i)^{2p-2} - (3-i)(-1+i)^{2p-2} + (-1)^{2p-2}4^{2p}\Big]\right] \pmod p \\
&\equiv 12\left[\frac{1}{5\cdot 4}\left[\frac{-3-i}{-1-i} - \frac{3-i}{-1+i} - 4\right]\right] + 8\left[\frac{1}{5\cdot 16}\Big[(-3-i) - (3-i) + 16\Big]\right] \pmod p \\
&\equiv 1 \pmod p
\end{aligned}$$

as desired.

Editorial note: By the same argument, one can prove that if $p > 5$ is congruent to 3 modulo 4, then the left side evaluates to $-12 \cdot 5^{-1} \pmod p$. For this, one follows the above argument except that for the last three lines of the proof one uses that $i = \sqrt{-1}$ is quadratic over $\mathbb{Z}_p$, therefore the Frobenius automorphism $x \mapsto x^p \pmod p$ sends $(-1-i)^p$ into $(-1+i)^p$ and vice versa.

Also solved partially by Dmitry Fleischman.

An inequality with sums of ratios involving Fibonacci numbers

H-816 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania
(Vol. 55, No. 4, November 2017)

Prove that for a positive integer n,

$$\frac{F_1}{(F_1^2+F_2^2)^2}+\frac{F_2}{(F_1^2+F_2^2+F_3^2)^2}+\cdots+\frac{F_n}{(F_1^2+F_2^2+\cdots+F_{n+1}^2)^2}\geq\frac{1}{F_{n+2}}-\frac{1}{F_{n+2}^2}.$$

Editorial note: Kenny B. Davenport pointed out that a more general version of this problem has already appeared as **B1178** in Volume 53, Number 3, August 2015, whose solution was published in Volume 54, Number 3, August 2016.

Also solved by Dmitry Fleischman, Wei Kai-Lai, and John Risher (jointly) and the proposers.