ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by March 15, 2009. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1040 Proposed by Paul S. Bruckman, Sointula, Canada

If $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the standard *Fibonomial coefficient*, prove the following identities, valid for $m = 0, 1, 2, \ldots$.

(a) $\displaystyle\sum_{k=0}^{2m} (-1)^k \begin{bmatrix} 4m+1 \\ 2k \end{bmatrix} F_{2k} = 0$

(b) $\displaystyle\sum_{k=0}^{2m+1} (-1)^k \begin{bmatrix} 4m+3 \\ 2k+1 \end{bmatrix} F_{2k+1} = 0$

(c) $\displaystyle\sum_{k=0}^{4m} (-1)^{k(k+1)/2} \begin{bmatrix} 4m \\ k \end{bmatrix} F_k = 0$

B-1041 Proposed by Paul S. Bruckman, Sointula, Canada

Prove that the following expression has a limit as $n \to \infty$, and find the limit.

$$\{(F_{n+1})^{1/2} + (F_n)^{1/2}\}/(F_{n+2})^{1/2}.$$

B-1042 Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universidad Politécnica de Cataluña, Barcelona, Spain

Let T_n be the nth triangular number defined by $T_n = \binom{n+1}{2}$ for all $n \geq 1$. Prove that

$$\frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{T_k}{F_k}\right)^2 \geq \frac{T_{n+1}^2}{9F_nF_{n+1}}.$$

B-1043 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad Politécnica de Cataluña, Barcelona, Spain

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers. Prove that

$$\left(\frac{F_nF_{n+1}}{n}\right)^n \geq \prod_{cyclic}\frac{\alpha_1F_1^2 + \cdots + \alpha_nF_n^2}{\alpha_1 + \cdots + \alpha_n} \geq (F_1F_2\cdots F_n)^2.$$

SOLUTIONS

A Square Root of a Pell Number Polynomial

B-1029 Proposed by José Luis Díaz-Barrero, Universitat Politécnica de Cataluñya, Barcelona, Spain (Vol. 45, no. 1, February 2007)

Let P_n be the nth Pell number. Prove that for all $n \geq 0$,
$$\left\{2\left(P_n^4+16P_{n+1}^4+P_{n+2}^4\right)\right\}^{1/2}$$
is a positive integer.

Solution by H.-J. Seiffert, Thorwaldsnstr. 13, Berlin, Germany

If a and b are any real numbers, then, as is easily verified by squaring and expanding both sides,
$$\{2(a^4+b^4+(a+b)^4)\}^{1/2}=2(a^2+ab+b^2).$$

Let n be any integer. Since $P_n+2P_{n+1}=P_{n+2}$, the desired result follows from the above identity when taking $a=P_n$ and $b=2P_{n+1}$.

Taking $a=F_n$ and $b=F_{n+1}$ and using the known relation $F_n^2+F_{n+1}^2=F_{2n+1}$, which is (I_{11}) in Hoggatt's list, one obtains
$$\{2(F_n^4+F_{n+1}^4+F_{n+2}^4)\}^{1/2}=2(F_{2n+1}+F_nF_{n+1}).$$

Also solved by Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, G. C. Greubel, Russell J. Hendel, and the proposer.

Pell and Fibonacci Numbers Equalities

B-1030 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 45, no. 1, February 2007)

Let P_n be the nth Pell number.

(a) Let r and s be integers and let $m=\gcd(P_r,F_s,P_{r-1}-F_{s-1})$. Prove that $F_nP_{n+r}\equiv P_nF_{n+s} \pmod m$ for all $n\in\mathbb{Z}$.

(b) Show that $F_nP_{n+8}\equiv P_nF_{n+18} \pmod{68}$ for all $n\in Z$.

(c) Find integers r and s, not both zero, such that $F_nP_{n+r}\equiv P_nF_{n+s} \pmod{13}$ for all $n\in Z$.

Solution by Paul S. Bruckman, P. O. Box 150, Sointula, BC V0N 3E0 (Canada)

Given integers r and s, the Pell and the Fibonacci sequences satisfy the following relations for all integers n:
$$P_{n+r}=P_{n+1}P_r+P_nP_{r-1}; \tag{1}$$
$$F_{n+s}=F_{n+1}F_s+F_nF_{s-1}. \tag{2}$$

Introduce the following definition:
$$D(n;r,s)\equiv F_nP_{n+r}-P_nF_{n+s}. \tag{3}$$

We see from (1) and (2) that $D(n;r,s) = F_n(P_{n+1}P_r + P_nP_{r-1}) - P_n(F_{n+1}F_s + F_nF_{s-1})$, or after simplification:

$$D(n;r,s) = F_nP_{n+1}P_r - P_nF_{n+1}F_s + F_nP_n(P_{r-1} - F_{s-1}). \qquad (4)$$

We see from (4) that if M is any common divisor of P_r, F_s, and $(P_{r-1} - F_{s-1})$, then $M \mid D(n;r,s)$; in particular, $m \mid D(n;r,s)$. This is equivalent to the statement of Part (a).

(b) Take $r = 8$ and $s = 18$. We find that $P_8 = 408 = 2^3 \cdot 3 \cdot 17$, $F_{18} = 2584 = 2^3 \cdot 17 \cdot 19$, and $P_7 - F_{17} = 169 - 1597 = -1428 = -2^2 \cdot 3 \cdot 7 \cdot 17$. In this case, $m = 2^2 \cdot 17 = 68$, showing that $68 \mid D(n;8,18)$, which is equivalent to the assertion of Part (b).

(c) Since we require $m = 13$, it follows that we must take $r = 7a$ and $s = 7b$ for some positive integers a and b. For $P_7 = 169 = 13^2$, and 13 does not divide any P_n for $1 \le n < 7$; on the other hand, $13 \mid P_{7a}$, $a = 1, 2, \ldots$. Thus, $13 \mid P_n$ if and only if $7 \mid n$. Likewise, $13 \mid F_n$ if and only if $7 \mid n$. These statements depend on the theory of the so-called "rank of appearance" of integers in linear second-order sequences.

We find that $F_6 = 8 \equiv -5 \pmod{13}$, $F_{13} = 233 \equiv -1 \pmod{13}$, $F_{20} = 6765 \equiv +5 \pmod{13}$; also, $P_6 = 70 \equiv +5 \pmod{13}$. We may therefore take $r = 7$, $s = 21$ (i.e., $a = 1$, $b = 3$). We see that this pair satisfies the required conditions: $P_7 = 13^2$, hence, $13 \mid P_7$; $F_{21} = 10946 = 13 \cdot 842$, hence, $13 \mid F_{21}$; and $P_6 - F_{20} = 70 - 6765 = -6695 = -13 \cdot 515$, hence, $13 \mid (P_6 - F_{20})$. Thus, $13 \mid D(n;7,21)$. Q.E.D.

Also solved by the proposer.

There Is a Limit!

B-1031 Proposed by Andrew Cusumano, Great Neck, NY (Vol. 45, no. 2, May 2007)

For any positive integer a, find an integer x such that

$$\lim_{n\to\infty}\left(\frac{F_{n+a}}{F_n}\right) = \frac{(2+\sqrt{5})F_a + F_{a-x}}{2}.$$

Solution by Rebecca A. Hillman, University of South Carolina Sumter, Sumter, SC 29150 and Steve Edwards, Southern Polytechnic State University, Marietta, GA (separately)

From the Binet form for the Fibonacci numbers it follows that $\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \alpha$, and a quick induction using the defining recurrence shows that $F_{n+a} = F_aF_{n+1} + F_{a-1}F_n$. It follows that

$$\lim_{n\to\infty}\left(\frac{F_{n+a}}{F_n}\right) = \lim_{n\to\infty}\left(\frac{F_aF_{n+1}+F_{a-1}F_n}{F_n}\right)$$

$$= F_a\alpha + F_{a-1}$$

$$= \frac{F_a(1+\sqrt{5})+2F_{a-1}}{2}$$

$$= \frac{F_a(2+\sqrt{5})+2F_{a-1}-F_a}{2}$$

$$= \frac{(2+\sqrt{5})F_a+F_{a-3}}{2}.$$

Therefore, $x = 3$.

Also solved by Paul S. Bruckman, Charles K. Cook, G. C. Greubel, Russell J. Hendel, Scott Duke Kominers (student), Harris Kwong, Jaroslav Seibert, H.-J. Seiffert, O. P. Sikltal, and the proposer.

<u>One Term is 1 More Than the Other</u>

<u>B-1032</u> Proposed by Andrew Cusumano, Great Neck, NY (Vol. 45, no. 2, May 2007)

Find a positive integer x such that

$$\frac{F_{n+3}(F_{n+1}F_{n+2}+F_n^2)(F_{n+2}F_{n+3}+F_{n+1}^2)-F_{n+1}^x}{F_{n+2}} - F_{n+1}F_{n+2}F_{n+3}(2F_{n+2}+F_n) = 1.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Every term, except the one involving F_{n+1}^x, on the left hand side, is of order 4, hence, it is natural to expect $x = 5$. To affirm our speculation, we will prove that $S = F_{n+2}$, where

$$S = F_{n+3}(F_{n+1}F_{n+2}+F_n^2)(F_{n+2}F_{n+3}+F_{n+1}^2) - F_{n+1}^5 - F_{n+1}F_{n+2}^2F_{n+3}(2F_{n+2}+F_n).$$

First, we find

$$F_{n+3}(F_{n+1}F_{n+2}+F_n^2) = (F_{n+2}+F_{n+1})[F_{n+2}F_{n+1}+(F_{n+2}-F_{n+1})^2]$$

$$= (F_{n+2}+F_{n+2})(F_{n+2}^2-F_{n+2}F_{n+1}+F_{n+1}^2)$$

$$= F_{n+2}^3+F_{n+1}^3.$$

Thus,

$$S = F_{n+2}^4 F_{n+3} + F_{n+1}^3 F_{n+2} F_{n+3} + F_{n+1}^2 F_{n+2}^3 - 2F_{n+1} F_{n+2}^3 F_{n+3} - F_n F_{n+1} F_{n+2}^2 F_{n+3}.$$

It follows from d'Ocagne's formula that

$$\begin{aligned}
&F_{n+2}^4 F_{n+3} + F_{n+1}^2 F_{n+2}^3 - 2F_{n+1} F_{n+2}^3 F_{n+3} \\
&= F_{n+2}^4 F_{n+3} - F_{n+1} F_{n+2}^3 F_{n+3} + F_{n+1}^2 F_{n+2}^3 - F_{n+1} F_{n+2}^3 F_{n+3} \\
&= F_{n+2}^3 F_{n+3}(F_{n+2} - F_{n+1}) + F_{n+1} F_{n+2}^3 (F_{n+1} - F_{n+3}) \\
&= F_n F_{n+2}^3 F_{n+3} - F_{n+1} F_{n+2}^4 \\
&= F_{n+2}^3 (F_n F_{n+3} - F_{n+1} F_{n+2}) \\
&= (-1)^{n+1} F_{n+2}^3.
\end{aligned}$$

Finally, apply Simson's formula to obtain

$$\begin{aligned}
F_{n+1}^3 F_{n+2} F_{n+3} - F_n F_{n+1} F_{n+2}^2 F_{n+3} &= F_{n+1} F_{n+2} F_{n+3} (F_{n+1}^2 - F_n F_{n+2}) \\
&= (-1)^{n+2} F_{n+1} F_{n+2} F_{n+3},
\end{aligned}$$

and

$$\begin{aligned}
S &= (-1)^{n+1} F_{n+2}^3 + (-1)^{n+2} F_{n+1} F_{n+2} F_{n+3} \\
&= (-1)^{n+2} F_{n+2} (F_{n+1} F_{n+3} - F_{n+2}^2) \\
&= (-1)^{n+2} F_{n+2} \cdot (-1)^{n+2} \\
&= F_{n+2}.
\end{aligned}$$

Therefore, the given identity holds when $x = 5$.

Also solved by Paul S. Bruckman, Charles K. Cook and Rebecca A. Hillman (jointly), G. C. Greubel, Russell J. Hendel, Jaroslav Seibert, H.-J. Seifert, and the proposer.

A Fibonacci Array

B-1033 Proposed by Jyoti P. Shiwalkar and M. N. Despande, Nagpur, India (Vol. 45, no. 2, May 2007)

Let i be a positive integer and $j = 1, 2, \dots, i$. Define a triangular array $a(i, j)$ satisfying the following three conditions.

(1) $a(i, 1) = F_i$ for all i.

(2) $a(i, i) = i$ for all i.

(3) $a(i, j) = a(i-1, j) + a(i-2, j) + a(i-1, j-1) - a(i-2, j-1)$.

For instance, if $i = 8$, then the array will be as follows:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & & & & & \\
 & & & & & 2 & & 2 & & 3 & & & & & \\
 & & & & 3 & & 5 & & 3 & & 4 & & & & \\
 & & & 5 & & 8 & & 9 & & 4 & & 5 & & & \\
 & & 8 & & 15 & & 15 & & 14 & & 5 & & 6 & & \\
 & 13 & & 26 & & 31 & & 24 & & 20 & & 6 & & 7 & \\
21 & & 46 & & 57 & & 54 & & 35 & & 27 & & 7 & & 8
\end{array}
$$

Find a closed form for $\displaystyle\sum_{j=1}^{i} a(i, j)$.

Solution by Michael R. Bacon and Charles K. Cook, emeritus, University of South Carolina Sumter, Sumter, SC 29150

First note that the following:

$$\sum_{j=1}^{1} a(1, j) = a(1, 1) = 1, \quad \sum_{j=1}^{2} a(2, j) = a(2, 1) + a(2, 2) = 1 + 2 = 3,$$

$$\sum_{j=1}^{3} a(3, j) = a(3, 1) + a(3, 2) + a(3, 3) = 2 + 2 + 3 = 7,$$

$$\sum_{j=1}^{4} a(4, j) = a(4, 1) + a(4, 2) + a(4, 3) + a(4, 4) = 3 + 5 + 3 + 4 = 15.$$

Thus, it can be conjectured that $\sum_{j=1}^{i} a(i,j) = 2^i - 1$. We prove this by induction.

In fact,

$$\sum_{j=1}^{i+1} a(i,j) = a(i+1,i+1) + a(i+1,1) + \sum_{j=2}^{i} a(i,j) = (i+1) + F_{i+1} + \sum_{j=2}^{i} a(i,j)$$

$$= (i+1) + F_i + \sum_{j=2}^{i} a(i,j) + \sum_{j=2}^{i} a(i-1,j) + \sum_{j=2}^{i} a(i,j-1) - \sum_{j=2}^{i} a(i-1,j-1).$$

Next it is seen, using the induction hypothesis, that

$$\sum_{j=2}^{i} a(i,j) = \sum_{j=1}^{i} a(i,j) - F_i = 2^i - 1 - F_i, \quad \sum_{j=2}^{i} a(i-1,j) = 2^{i-1} - 1 - F_{i-1}$$

$$\sum_{j=2}^{i} a(i,j-1) = 2^i - (i+1) \quad \text{and} \quad \sum_{j=2}^{i} a(i-1,j-1) = 2^{i-1} - 1.$$

Hence,

$$\sum_{j=1}^{i+1} a(i+1,j) = (i+1) + F_{i+1} + (2^i - 1 - F_i) + (2^{i-1} - 1 - F_{i-1}) + (2^i - (i+1)) - (2^{i-1} - 1)$$

$$= 2^i + 2^i - 1 = 2^{i+1} - 1.$$

Thus, by induction, the conjecture holds for all positive integral i.

Also solved by Paul S. Bruckman, G. C. Greubel, Russell J. Hendel, Harris Kwong, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

We would like to acknowledge that a late solution to Problem B-1028 by G. C. Greubel was received. Also, Rebecca A. Hillman was a co-solver of Problem B-1019.