ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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PROBLEMS PROPOSED IN THIS ISSUE

H-631 Proposed by Jayantibhai M. Patel, Ahmedabad, India

For any positive integer $n \geq 2$, prove that the value of the following determinant

$$\begin{vmatrix} (F_nF_{n+2}+F_{n+1}^2) & F_n^2 & F_{n+1}^2 & F_{n+2}^2 & -(F_nF_{n+2}+F_{n+1}^2) \\ (F_nF_{n+2}+F_{n+1}^2) & F_nF_{n+3} & -F_{n+1}L_{n+1} & F_{n-1}F_{n+2} & (F_nF_{n+2}+F_{n+1}^2) \\ 0 & 2F_{n+1}F_{n+2} & 2F_nF_{n+2} & -2F_nF_{n+1} & 0 \\ (F_nF_{n+2}+F_{n+1}^2) & -F_nF_{n+3} & F_{n+1}L_{n+1} & -F_{n-1}F_{n+2} & (F_nF_{n+2}+F_{n+1}^2) \\ -(F_nF_{n+2}+F_{n+1}^2) & F_n^2 & F_{n+1}^2 & F_{n+2}^2 & (F_nF_{n+2}+F_{n+1}^2) \end{vmatrix}$$

is $-(2(F_nF_{n+2}+F_{n+1}^2))^5$.

H-632 Proposed by Paul S. Bruckman, Sointula, Canada

Prove the following identities:

1. $$1+\sum_{n=1}^{\infty}(-1)^n\frac{5^{-n/2}\alpha^{n(3n-1)/2}}{F_1F_2\dots F_n}=\prod_{n=0}^{\infty}\{1+4(-1)^n\alpha^{-10n-5}-\alpha^{-20n-10}\}^{-1}.$$

2. $$1+\sum_{n=1}^{\infty}(-1)^n\frac{5^{-n/2}\alpha^{n(3n+1)/2}}{F_1F_2\dots F_n}=\prod_{n=0}^{\infty}\{1-(-1)^n\alpha^{-10n-5}-\alpha^{-20n-10}\}^{-1}.$$

Here, α is the golden section.

H-633 Proposed by Kenneth B. Davenport, Dallas, PA

Let $A,\ B$ and C be $A=\sum_{n=0}^{\infty}(-1)^n\Big(\frac{1}{7n+1}+\frac{1}{7n+6}\Big)$, $B=\sum_{n=0}^{\infty}(-1)^n\Big(\frac{1}{7n+2}+\frac{1}{7n+5}\Big)$ and $C=\sum_{n=0}^{\infty}(-1)^n\Big(\frac{1}{7n+3}+\frac{1}{7n+4}\Big)$. Show that $A+B=C$.

H-634 Proposed by Ovidiu Furdui, Kalamazoo, MI

Prove that $\displaystyle\sum_{j=0}^{\lfloor (n-1)/2\rfloor} \frac{1}{n-2j}\binom{n-j-1}{j} = \frac{\alpha^n-1}{n} + \frac{\beta^n-(-1)^n}{n}$ holds for all $n \geq 1$,

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

SOLUTIONS

Fibonacci polynomials and trigonometric functions

**H-617 Proposed by H.-J. Seiffert, Berlin, Germany
(Vol. 42, no. 4, November 2004)**

The sequence of Fibonacci polynomials is defined by $F_0(x) = 0,\ F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \geq 0$. Show that, for all real numbers x and all nonnegative integers n,

$$\textbf{(a)}\ \sum_{k=0}^{2n}(-1)^{\lfloor k/2\rfloor}\binom{2n}{k}F_k(x) = \sqrt{2}(-1)^n(x^2+4)^{n/2}F_n(x)\cos\Big(ny+\frac{\pi}{4}\Big),$$

$$\textbf{(b)}\ \sum_{k=0}^{2n}(-1)^{\lceil k/2\rceil}\binom{2n}{k}F_k(x) = \sqrt{2}(-1)^n(x^2+4)^{n/2}F_n(x)\sin\Big(ny+\frac{\pi}{4}\Big),$$

where $y = \arccos\dfrac{x}{\sqrt{x^2+4}}$. Here, $\lfloor . \rfloor$ and $\lceil . \rceil$ denote the floor and ceiling function, respectively.

Solution by the proposer

It is known that $F_j(x) = (\alpha(x)^j - \beta(x)^j)/\sqrt{x^2+4}$, where $\alpha_j(x) = (x+\sqrt{x^2+4})/2$ and $\beta_j(x) = (x-\sqrt{x^2+4})/2$. If $i = \sqrt{-1}$, then, by the Binomial Theorem,

$$S_n(x) := \sum_{j=0}^{2n}(-i)^j\binom{2n}{j}F_j(x) = \frac{1}{\sqrt{x^2+4}}\sum_{j=0}^{2n}\binom{2n}{j}\big((-i\alpha(x))^j - (-i\beta(x))^j\big),$$

so

$$S_n(x) = \frac{(1-i\alpha(x))^{2n} - (1-i\beta(x))^{2n}}{\sqrt{x^2+4}}.$$

Since $1-\alpha(x)^2 = -x\alpha(x)$ and $1-\beta(x)^2 = -x\beta(x)$, we have that $(1-i\alpha(x))^2 = -\alpha(x)(x+2i)$ and $(1-i\beta(x))^2 = -\beta(x)(x+2i)$. It follows that $S_n(x) = (-1)^nF_n(x)(x+2i)^n$. Using $\cos y = x/\sqrt{x^2+4}$, $\sin y = 2/\sqrt{x^2+4}$, and Euler's relation $e^{iy} = \cos y + i\sin y$, we then have

$$S_n(x) = (-1)^n(x^2+4)^{n/2}F_n(x)e^{iny}.$$

Equating the real and imaginary parts give

$$U_n(x) := \mathrm{Re}S_n(x) = \sum_{k=0}^{n}(-1)^k\binom{2n}{2k}F_{2k}(x) = (-1)^n(x^2+4)^{n/2}F_n(x)\cos(ny),$$

and

$$V_n(x) := \mathrm{Im}S_n(x) = \sum_{k=0}^{n-1}(-1)^{k+1}\binom{2n}{2k+1}F_{2k+1}(x) = (-1)^n(x^2+4)^{n/2}F_n(x)\sin(ny).$$

Obviously, the sum of (a) equals $U_n(x) - V_n(x)$, while the sum of (b) is $U_n(x) + V_n(x)$, so that the desired identities follow from the addition laws of the cosine and the sine.

Example. Since $\arccos(1/\sqrt{2}) = \pi/4$, with $x = 2$ we obtain

$$\sum_{k=0}^{2n}(-1)^{\lfloor k/2\rfloor}\binom{2n}{k}P_k = (-1)^n 2^{(3n+1)/2}P_n\cos\Big(\frac{(n+1)\pi}{4}\Big),$$

and

$$\sum_{k=0}^{2n}(-1)^{\lceil k/2\rceil}\binom{2n}{k}P_k = (-1)^n 2^{(3n+1)/2}P_n\sin\Big(\frac{(n+1)\pi}{4}\Big),$$

where $P_k = F_k(2)$ is the kth Pell number.

Also solved by Paul S. Bruckman and Kenneth B. Davenport.

The Exponential Function Revisited

H-618 Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade (Vol. 43, no. 1, February 2005)

Prove that there exists a constant $c \geq 2.5$ such that the inequality

$$e^x \geq 1 + x^\alpha$$

holds for each $x \geq 0$ if and only if $\alpha \in [1, c]$. What is the value of c?

Solution by the proposer

It is clear that if $\alpha \leq 0$, then the stated inequality is false if x is a small positive real number (as e^x tends to 1 when $x > 0$ tends to zero, while $x^\alpha \geq 1$). It is also false when

$0 < \alpha < 1$ at the point $x_0 = \exp(1/(\alpha - 1)) > 0$ since for this value of x_0 we have $x_0 < 1$, therefore

$$e^{x_0} - 1 \le x_0 e^{x_0} < e x_0 = x_0^{\alpha}.$$

For $\alpha = 1$, the inequality becomes the familiar one $e^x \ge 1 + x$. Finally, we show that the inequality holds with $\alpha = 2.5$. If $x \in [0, 1]$, then $e^x - 1 \ge x \ge x^{2.5}$. If $x > 1$, then

$$e^x - 1 - x^{2.5} = x\Big(\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} - x^{1.5}\Big) > x\Big(1 + \frac{x}{2} + \frac{x^2}{6} + x^3\Big(\sum_{n\ge 4} \frac{1}{n!}\Big) - x^{1.5}\Big).$$

By the AGM inequality, $x/2 + x^2/6 \ge x^{1.5}/\sqrt{3}$. Hence,

$$e^x - 1 - x^{2.5} > x\Big(1 - \Big(1 - \frac{1}{\sqrt{3}}\Big)x^{1.5} + x^3\Big(e - \frac{8}{3}\Big)\Big) = x\Big(\Big(1 - \frac{(3-\sqrt{3})x^{1.5}}{6}\Big)^2 + x^3\Big(e - 3 + \frac{\sqrt{3}}{6}\Big)\Big),$$

and since $e - 3 + \dfrac{\sqrt{3}}{6} = 0.006\ldots > 0$, the proof is complete.

Comment. Looking at the graph of $y = e^x - 1 - x^{\alpha}$, we see that the best value of c is the one for which the tangent at $\alpha = c$ is the x-axis. Hence, c is the common solution of the two equations $e^x - 1 - x^c = 0$ and $e^x - c x^{c-1} = 0$, which is $c = 2.632748338\ldots$.

Also solved by Kenneth Davenport.

A 5×5 Determinant

H-619 Proposed by Jayantibhai M. Patel, Ahmedabad, India (Vol. 43, no. 1, February 2005)

For any positive integer $n \ge 2$, prove that the value of the following determinant

$$\begin{vmatrix} -(6F_{n-1}^2 - L_n^2) & 2F_{n+1}F_{n+2} & 2F_nF_{n+2} & 2F_{n-1}F_{n+2} & 2F_{n-2}F_{n+2} \\ 2F_{n+2}F_{n+1} & -(2F_{n-1}^2 + 5F_n^2) & 2F_nF_{n+1} & 2F_{n-1}F_{n+1} & 2F_{n-2}F_{n+1} \\ 2F_{n+2}F_n & 2F_{n+1}F_n & -(4F_n^2 + L_n^2) & 2F_{n-1}F_n & 2F_{n-2}F_n \\ 2F_{n+2}F_{n-1} & 2F_{n+1}F_{n-1} & 2F_nF_{n-1} & -(5F_n^2 + 2F_{n+1}^2) & 2F_{n-2}F_{n-1} \\ 2F_{n+2}F_{n-2} & 2F_{n+1}F_{n-2} & 2F_nF_{n-2} & 2F_{n-1}F_{n-2} & -(6F_{n+1}^2 - F_n^2) \end{vmatrix}$$

is $(6F_n^2 + L_n^2)^5$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Let A be the underlying matrix of the given determinant. We shall find its eigenvectors, and show that the eigenvalues are λ with multiplicity one and $-\lambda$ with multiplicity four, where $\lambda = 6F_n^2 + L_n^2$.

First, we need to derive two identities, both of which rely on the following result:

$$2F_{n+1}^2 + 2F_{n-1}^2 = (F_{n+1} - F_{n-1})^2 + (F_{n+1} + F_{n-1})^2 = F_n^2 + L_n^2.$$

Together with $L_n = F_{n+1} + F_{n-1} = F_n + 2F_{n-1}$, we find that

$$\begin{aligned} 2F_{n+2}^2 + 6F_{n-1}^2 - L_n^2 &= 2(F_{n+1} + F_n)^2 + 6F_{n-1}^2 - (F_n + 2F_{n-1})^2 \\ &= 2(F_{n+1}^2 + F_{n-1}^2) + F_n^2 + 4F_n(F_{n+1} - F_{n-1}) = 6F_n^2 + L_n^2. \end{aligned} \tag{1}$$

Likewise, since $L_n = F_{n+1} + F_{n-1} = 2F_{n+1} - F_n$, we also have

$$\begin{aligned} 2F_{n-2}^2 + 6F_{n+1}^2 - L_n^2 &= 2(F_n - F_{n-1})^2 + 6F_{n+1}^2 - (2F_{n+1} - F_n)^2 \\ &= F_n^2 + 2(F_{n-1}^2 + F_{n+1}^2) + 4F_n(F_{n+1} - F_{n-1}) = 6F_n^2 + L_n^2. \end{aligned} \tag{2}$$

Using identities (1) and (2), we can write

$$A = -\lambda I + 2\boldsymbol{u}\boldsymbol{u}^T,$$

where $\boldsymbol{u}$ is the vector $(F_{n+2}, F_{n+1}, F_n, F_{n-1}, F_{n-2})^T$. Note that

$$\begin{aligned} \boldsymbol{u}^T\boldsymbol{u} = F_{n+2}^2 + F_{n+1}^2 + F_n^2 + F_{n-1}^2 + F_{n-2}^2 &= (F_{n+1} + F_n)^2 + F_{n+1}^2 + F_n^2 + F_{n-1}^2 + (F_n - F_{n-1})^2 \\ &= 2(F_{n+1}^2 + F_{n-1}^2) + 3F_n^2 + 2F_n(F_{n+1} - F_{n-1}) = 6F_n^2 + L_n^2. \end{aligned}$$

Hence,

$$A\boldsymbol{u} = (-\lambda I + 2\boldsymbol{u}\boldsymbol{u}^T)\boldsymbol{u} = -\lambda\boldsymbol{u} + 2\lambda\boldsymbol{u} = \lambda\boldsymbol{u};$$

in other words, $\boldsymbol{u}$ is an eigenvector of A corresponding to the eigenvalue λ. For $n \geq 3$, define the vectors

$$\begin{aligned} \boldsymbol{v}_1 &= (F_{n-2}, 0, 0, 0, -F_{n+2})^T, \\ \boldsymbol{v}_2 &= (0, F_{n-2}, 0, 0, -F_{n+1})^T, \\ \boldsymbol{v}_3 &= (0, 0, F_{n-2}, 0, -F_n)^T, \\ \boldsymbol{v}_4 &= (0, 0, 0, F_{n-2}, -F_{n-1})^T; \end{aligned}$$

but for $n = 2$, define

$$\begin{aligned} \boldsymbol{v}_1 &= (1, 0, 0, -3, 0)^T, \\ \boldsymbol{v}_2 &= (0, 1, 0, -2, 0)^T, \\ \boldsymbol{v}_3 &= (0, 0, 1, -1, 0)^T, \\ \boldsymbol{v}_4 &= (0, 0, 0, 0, 1)^T. \end{aligned}$$

It is easy to verify that $\boldsymbol{u}^T\boldsymbol{v}_i = 0$ for each i. Hence,

$$A\boldsymbol{v}_i = (-\lambda I + 2\boldsymbol{u}\boldsymbol{u}^T)\boldsymbol{v}_i = -\lambda\boldsymbol{v}_i;$$

consequently, $\boldsymbol{v}_i$ are the eigenvectors of A corresponding to the eigenvalue $-\lambda$. It is obvious that $\{\boldsymbol{u}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is an independent set, therefore the determinant of A is λ^5.

Correction. In the statement of this problem (The Fibonacci Quarterly **43.1** (2005): 91), the entry $(5,4)$ of the determinant was erroneously written as "$2F_{n-1}F_{n+2}$" instead of "$2F_{n-1}F_{n-2}$".

Also solved by Paul S. Bruckman.

A Trigonometric Inequality

H-620 Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Ciaurri Ramírez, Logroño, Spain
(Vol. 43, no. 1, February 2005)

Let ABC be a triangle. Prove that the following inequality holds for $\alpha \in [0, \pi/2)$:

$$\sqrt{F_{n+1}F_{n+2}}\cos(C-\alpha) + \sqrt{F_{n+2}F_n}\cos(B-\alpha) + \sqrt{F_nF_{n+1}}\cos(A-\alpha) \leq 2F_{n+2}\cos\left(\frac{\pi}{3}-\alpha\right).$$

Solution by H.-J. Seiffert, Berlin, Germany

If u, v and w are any nonnegative real numbers, then (see [1], inequality (5.2) on page 424 and inequality (6.6) on page 428)

$$\sqrt{uv}\cos C + \sqrt{vw}\cos B + \sqrt{wu}\cos A \leq \frac{1}{2}(u+v+w),$$

and

$$\sqrt{uv}\sin C + \sqrt{vw}\sin B + \sqrt{wu}\sin A \leq \frac{\sqrt{3}}{2}(u+v+w).$$

Multiplying the first inequality by $\cos\alpha$ and the second by $\sin\alpha$ and adding the resulting inequalities gives

$$\sqrt{uv}\cos(C-\alpha) + \sqrt{vw}\cos(B-\alpha) + \sqrt{wu}\cos(A-\alpha) \leq (u+v+w)\cos\left(\frac{\pi}{3}-\alpha\right),$$

where we have used the known trigonometric relations $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, and the addition formula for the cosine. Taking $u = F_{n+2}$, $v = F_{n+1}$ and $w = F_n$, where $n \geq -1$ yields the desired inequality.

[1] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, "Classical and New Inequalities in Analysis", Kluwer Academic Publishers, Dordrecht, 1993.

Also solved by Paul S. Bruckman, G. C. Greubel, Ovidiu Furdui and the proposers.

Late Acknowledgement. Kenneth B. Davenport solved H-613.

Errata. In H-629, the inequality "< 0.5" should be "≤ 0.5".