ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2005. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-996 Proposed by Paul S. Bruckman, Canada

Prove the following congruences, for all integers n:

(1) $L_n \equiv 30^n + 50^n \pmod{79}$;

(2) $L_n \equiv 10^n + 80^n \pmod{89}$.

B-997 Proposed by Br. J. Mahon, Australia

Prove that

$$\sum_{i=1}^{n} \frac{L_{i-3} - 5(-1)^i}{(L_i+1)(L_{i+1}+1)} = \frac{3}{2} - \frac{L_n+1}{L_{n+1}+1}$$

for all $n \geq 1$.

B-998 Proposed by José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Barcelona, Spain

Let n be a positive integer and let $F_n,\ L_n$ and P_n be respectively the n^{th} Fibonacci, Lucas and Pell number. Prove that

$$\frac{\left|\frac{|F_n-L_n|}{F_{2n}}+\frac{2F_{n+1}}{F_{2n}}-\frac{2}{P_n}\right|+\frac{|F_n-L_n|}{F_{2n}}+\frac{2F_{n+1}}{F_{2n}}+\frac{2}{P_n}}{\max\left\{\frac{1}{F_n},\frac{1}{L_n},\frac{1}{P_n}\right\}}$$

is an integer and determine its value.

B-999 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

Prove that

$$e^{2\sum_{k=1}^{n}\frac{F_{k-1}}{F_{k+2}}} \le F_{n+1} \le e^{\sum_{k=1}^{n}\frac{F_{k-1}}{\sqrt{F_kF_{k+1}}}}$$

for all $n \ge 1$.

B-1000 Proposed by Mihály Bencze, Romania

Prove that $F_{nF_n^k}$ is divisible by F_n^{k+1} for all $n \ge 1$ and $k \ge 1$.

SOLUTIONS

A Simplified Sum

B-981 Proposed by Steve Edwards, Southern Polytechnic State University Marietta, GA
(Vol. 42, no. 3, August 2004)

For all positive integers n, prove that

$$\sum_{k=1}^{n} F_{6k} = F_{6n+5} - \frac{5}{4}F_{3n+3}^2 + \frac{(-1)^n - 1}{2}.$$

Solution by Maitland A. Rose, University of South Carolina, Sumter, SC

Use is made of the identities $L_h - F_h = 2F_{h-1}$ and $L_{2h} = 5F_h^2 + 2(-1)^h$ and the fact that $(\alpha^6 - 1) = 8 + 4\sqrt{5}$, $(1 - \beta^6) = -8 + 4\sqrt{5}$.

$$\sum_{k=1}^{n} F_{6k} = \sum_{k=1}^{n} \frac{\alpha^{6k} - \beta^{6k}}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\alpha^6(\alpha^{6n} - 1)}{\alpha^6 - 1} - \frac{\beta^6(1 - \beta^{6n})}{1 - \beta^6}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{-8(\alpha^{6n+6} - \beta^{6n+6}) + 4\sqrt{5}(\alpha^{6n+6} + \beta^{6n+6}) + 8(\alpha^6 - \beta^6) - 4\sqrt{5}(\alpha^6 + \beta^6)}{16}\right]$$

$$= \left[\frac{-8F_{6n+6} + 4L_{6n+6} + 8F_6 - 4L_6}{16}\right]$$

$$= \left[\frac{8(L_{6n+6} - F_{6n+6}) - 4L_{6n+6} - 8}{16}\right]$$

$$= \frac{16F_{6n+5} - 4(5F_{3n+3}^2 + 2(-1)^{3n+3}) - 8}{16}$$

$$= F_{6n+5} - \frac{5}{4}F_{3n+3}^2 + \frac{(-1)^n - 1}{2}.$$

H.-J. Seiffert noticed the more general identity for $j \geq 2$, $\sum_{k=1}^{n} F_{jk} = F_{jn+j-1} - \frac{F_{j-1}-1}{L_j-2}L_{jn+j} - \frac{F_j}{L_j-2}$ *where* $n \in \mathbb{N}$.

Also solved by Paul S. Bruckman, Kenneth Davenport, Ovidiu Furdui, G.C. Greubel, Russell J. Hendel, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

Odd Sums!

B-982 Proposed by Harris Kwong, SUNY Fredonia, Fredonia NY (Vol. 42, no. 3, August 2004)

For odd positive integers k, evaluate the sums

$$\sum_{n=2}^{\infty} \frac{F_{kn}}{F_{k(n-1)}F_{k(n+1)}} \text{ and } \sum_{n=2}^{\infty} \frac{L_{kn}}{L_{k(n-1)}L_{k(n+1)}}.$$

Solution by H.-J. Seiffert, Berlin, Germany

Since k is odd, we have [see A.F. Horadam & Bro. J.M. Mahon. "Pell and Pell-Lucas Polynomials" *The Fibonacci Quarterly* **23.1** (1985): 7-20, eqns. (3.24) and (3.25)]

$$F_{k(n+1)} - F_{k(n-1)} = L_k F_{kn} \quad \text{and} \quad L_{k(n+1)} - L_{k(n-1)} = L_k L_{kn}.$$

Hence, for $n \geq 2$,

$$\frac{F_{kn}}{F_{k(n-1)}F_{k(n+1)}} = \frac{1}{L_k}\left(\frac{1}{F_{k(n-1)}} - \frac{1}{F_{k(n+1)}}\right)$$

and

$$\frac{L_{kn}}{L_{k(n-1)}L_{k(n+1)}} = \frac{1}{L_k}\left(\frac{1}{L_{k(n-1)}} - \frac{1}{L_{k(n+1)}}\right).$$

Let S_k and T_k denote the sums in question. Summing the above equations over $n = 2, 3, 4, \ldots$, we get, by telescoping,

$$S_k = \frac{1}{L_k}\left(\frac{1}{F_k} + \frac{1}{F_{2k}}\right) \quad \text{and} \quad T_k = \frac{1}{L_k}\left(\frac{1}{L_k} + \frac{1}{L_{2k}}\right).$$

Also solved by Paul S. Bruckman, Kenneth Davenport, Ovidiu Furdui, G.C. Greubel, Don Redmond, Jaroslav Seibert, and the proposer.

Integral Solutions to a Nonlinear System

B-983 Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain (Vol. 42, no. 3, August 2004)

Let n be a nonnegative integer. Prove that the system of equations

$$\frac{1}{2}(x + y + z) = F_{n+2},$$

$$\frac{1}{2}(x^2 + y^2 + z^2) = F_{n+2}^2 - F_n F_{n+1},$$

$$\frac{1}{2}(x^3 + y^3 + z^3) = F_{n+2}^3 - \frac{3}{2}F_n F_{n+1} F_{n+2}$$

has only integer solutions and determine them.

Solution by Paul S. Bruckman, Sointula, BC Canada, and Jaroslav Seibert, University Hradec Kralove, The Czech Republic (independently)

Let us denote by x, y, z the roots of a cubic equation with unknown u. Then,

$$(u - x)(u - y)(u - z) = u^3 - (x + y + z)u^2 + (xy + xz + yz)u - xyz = 0.$$

Computations using the given inequalities imply

$$x + y + z = 2F_{n+2},$$

$$xy + xz + yz = \frac{1}{2}[(x + y + z)^2 - (x^2 + y^2 + z^2)]$$

$$= \frac{1}{2}(4F_{n+2}^2 - 2F_{n+2}^2 + 2F_nF_{n+1}) = F_{n+2}^2 + F_nF_{n+1},$$

$$xyz = \frac{1}{6}(x + y + z)^3 - \frac{1}{2}(x + y + z)(x^2 + y^2 + z^2) + \frac{1}{3}(x^3 + y^3 + z^3)$$

$$= \frac{4}{3}F_{n+2}^3 - F_{n+2}(2F_{n+2}^2 - 2F_{n+1}F_n) + \frac{2}{3}F_{n+2}^3 - F_nF_{n+1}F_{n+2}$$

$$= F_nF_{n+1}F_{n+2}.$$

It follows that the numbers x, y, and z have to be the roots of the cubic equation

$$u^3 - 2F_{n+2}u^2 + (F_{n+2}^2 + F_nF_{n+1})u - F_nF_{n+1}F_{n+2} = 0.$$

However, it is easy to see that this equation can be written in the form

$$(u - F_n)(u - F_{n+1})(u - F_{n+2}) = 0.$$

This shows that $x = F_{n+2},\ y = F_{n+1},\ z = F_n$ (or any permutation thereof).

Also solved by Kenneth Davenport, Ovidiu Furdui, G.C. Greubel, H. -J. Seiffert, and the proposer.

A Lucas-Fibonacci-Diophantine Connection

B-984 Proposed by Juan Pla, Paris, France (Vol. 42, no. 3, August 2004)

Find solutions related to Lucas and Fibonacci numbers to the Diophantine equation

$$x^2 - 5y^2 + 2z^2 = \pm 1.$$

Solution by Paul S. Bruckman, Sointula, BC Canada

We recall the following identity: $(L_n)^2 - 5(F_n)^2 = 4(-1)^n$. Also, we recall that $2|L_n,\ 2|F_n$ iff $3|n$. Thus, we tentatively try $x = \frac{1}{2}L_{3n},\ y = \frac{1}{2}F_{3n}$, in which case, we obtain: $(-1)^n + 2z^2 = \pm 1$. This leads to z assuming the possible values $-1,\ 0$ or $+1$, depending on the parity of n. To avoid trivial variations in sign, we may confine ourselves to non-negative values of $x,\ y$ and z. Completing the analysis, we obtain the following solutions of the equation:

$$(x, y, z) = \left(\frac{1}{2}L_{6n}, \frac{1}{2}F_{6n}, 0\right), \left(\frac{1}{2}L_{6n+3}, \frac{1}{2}F_{6n+3}, 0\right), \left(\frac{1}{2}L_{6n+3}, \frac{1}{2}F_{6n+3}, 1\right), n = 0, 1, \cdots.$$

G.C. Greubel used a method based on a pell equation to give the general solution $x = \frac{1}{2}L_{3n}$, $y = \frac{1}{4}F_{3n}\left(c^{2m} + d^{2m}\right)$, $z = \frac{\sqrt{5}}{4\sqrt{2}}F_{3n}\left(c^{2m} - d^{2m}\right)$ *where* $c = 3 + \sqrt{10}, d = 3 - \sqrt{10}$, *and* $m = 0, \pm 1, \pm 2, \ldots$. *The separate solutions are given upon the choice of* n *being even or odd.*

Also solved by G.C. Greubel, H. -J. Seiffert, and the proposer.

Two Recurrence Relations

B-985 Proposed by Mario Catalani, University of Torino, Torino, Italy (Vol. 42, no. 3, August 2004)

Let $P_0 = 0, P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Define $U_n = F_{p_n}$ and $V_n = L_{p_n}$. For $n \geq 0$, show that

(a) $U_{n+2} = \frac{1}{2}\left(U_n(V_{n+1}^2 - 2(-1)^{n+1}) + U_{n+1}V_{n+1}\sqrt{5U_n^2 + 4(-1)^n}\right)$ with $U_0 = 0$ and $U_1 = 1$;

(b) $V_{n+2} = \frac{1}{2}\left(V_n(V_{n+1}^2 - 2(-1)^{n+1}) + U_{n+1}V_{n+1}\sqrt{5V_n^2 - 20(-1)^n}\right)$ with $V_0 = 2$ and $V_1 = 1$.

Solution by Paul S. Bruckman, Sointula, BC Canada

A simple induction argument shows that $P_k \equiv k \pmod 2$ for $k \geq 0$; hence $(-1)^{Pn} = (-1)^n$. Also, recall the following identities: $(L_n)^2 - 5(F_n)^2 = 4(-1)^n; F_nL_n = F_{2n}; (L_n)^2 -$

$2(-1)^n = L_{2n}$. These facts imply the following: $\{5(U_n)^2 + 4(-1)^n\}^{\frac{1}{2}} = V_n$; $\{5(V_n)^2 - 20(-1)^n\}^{\frac{1}{2}} = 5U_n$; $(V_{n+1})^2 - 2(-1)^{n+1} = L_{2P_{n+1}}$; $U_{n+1}V_{n+1} - F_{2P_{n+1}}$.

Therefore, the expression in the right member of (a) becomes:

$$\frac{1}{2}\left\{F_{P_n}L_{2P_{n+1}} + F_{2P_{n+1}}L_{P_n}\right\} = F_{2P_{n+1}+P_n} = F_{P_{n+2}} = U_{n+2}.$$

The expression in the right member of (b) becomes:

$$\frac{1}{2}\left\{L_{P_n}L_{2P_{n+1}} + 5F_{2P_{n+1}}F_{P_n}\right\} = L_{2P_{n+1}+P_n} = L_{P_{n+2}} = V_{n+2}.$$

Also solved by Kenneth Davenport, Ovidui Furdui, G.C. Greubel, H. -J. Seiffert, and the proposer.

We would like to acknowledge the receipt of late solutions to problems B-971 and B-975 from Kenneth Davenport.