A GENERALIZATION OF SEMI-COMPLETENESS FOR INTEGER SEQUENCES

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Notation

In this article, the notation $f_i(\infty)$ will be used to signify

$$\{ f_i \}_1^\infty .$$

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Given a sequence $f_i(\infty)$ of positive integers and two auxiliary sequences $k_i(\infty)$ of positive integers and $m_i(\infty)$ of nonnegative integers, we wish to consider the possibility of expanding an arbitrary positive integer n in the form

$$n = \sum_1^M \alpha_i f_i ,$$

where M is finite and each α_i is an integer (zero and negative values allowed) satisfying

$$-m_i \leq \alpha_i \leq k_i \quad \text{for } i = 1, 2, \ldots, M.$$

Throughout the paper, the convention is adopted that $k_i(\infty)$ and $m_i(\infty)$ will always denote given sequences of positive and nonnegative integers, respectively.

As an application of the results to be proved, we shall show that every positive integer n has an expansion in the form

$$n = \sum_1^M \alpha_i F_i^p ,$$

where p is a fixed integer greater than or equal to 2, $F_i(\infty) = \{1, 1, 2, 3, 5, \ldots\}$ is the usual Fibonacci sequence and α_i is an integer satisfying $|\alpha_i| \leq 2^{p-2}$ for each value of i.

DEFINITION 1: A sequence of positive integers $f_i(\infty)$, is said to be quasi-complete with respect to the sequences $k_i(\infty)$ and $m_i(\infty)$ iff (if and only if)

$$0 < n < 1 + \sum_1^N k_i f_i \quad \text{implies}$$

(1) $$n = \sum_1^N \alpha_i f_i \quad \text{with } \alpha_i \text{ integral and}$$

(2) $$-m_i \leq \alpha_i \leq k_i \quad \text{for} \quad i = 1, 2, \ldots, N.$$

The purpose of the present paper is to obtain a characterization of quasi-completeness and to investigate the conditions under which the representation in (1) is unique. Moreover, we will also show that any nondecreasing sequence of positive integers $f_i(\infty)$ which is either complete or semi-complete must also be quasi-complete.

Before proceeding to the proof of the characterization theorem, we recall some pertinent definitions and a lemma.

DEFINITION 2: (Reference 1). A sequence of positive integers $f_i(\infty)$ is complete iff every positive integer n has a representation in the form

(3) $$n = \sum_1^\infty c_i f_i, \quad \text{where each } c_i \text{ is either zero}$$

or one.

DEFINITION 3: (Reference 2). A sequence of positive integers $f_i(\infty)$ is semi-complete with respect to the sequence of positive integers $k_i(\infty)$, iff every positive integer n has a representation

(4) $$n = \sum_{1}^{\infty} c_i f_i \ ,$$ where each c_i is a nonnegative integer satisfying

(5) $$0 \leq c_i \leq k_i \ .$$

LEMMA 1: (Alder, Ref. 2, pp. 147-8). Let $f_i(\infty)$ be a given sequence of positive integers with $f_1 = 1$ and such that

$$f_{p+1} \leq 1 + \sum_{1}^{p} k_i f_i \quad \text{for } p = 1, 2, 3, \ldots,$$

where $k_i(\infty)$ is a fixed sequence of positive integers. Then for any positive integer n satisfying the inequality

$$0 < n < 1 + \sum_{1}^{N} k_i f_i \ ,$$

there exist nonnegative integers, $\alpha_i(N)$, such that

$$n = \sum_{1}^{N} \alpha_i f_i \quad \text{and} \quad 0 \leq \alpha_i \leq k_i$$

for i 1, 2, ..., N.

The following theorem gives a necessary and sufficient condition for quasi-completeness.

THEOREM 1: For given sequences, $k_i(\infty)$ and $m_i(\infty)$, the sequence of positive integers $f_i(\infty)$ with $f_1 = 1$ is quasi-complete iff

(6) $$f_{p+1} \leq 1 + \sum_{1}^{p} (k_i + m_i) f_i \quad \text{for } p = 1, 2, 3 \ldots$$

PROOF. Assume condition (6) is satisfied, and let n be a fixed positive integer satisfying

$$0 < n < 1 + \sum_1^N k_i f_i . \quad \text{Then}$$

$$0 < n + \sum_1^N m_i f_i < 1 + \sum_1^N (k_i + m_i) f_i ,$$

and Lemma 1 implies

(7) $$n + \sum_1^N m_i f_i = \sum_1^N \beta_i f_i ,$$ where each β_i is a non-negative integer satisfying $0 \leq \beta_i \leq k_i + m_i$ for each $i = 1, 2, \ldots, N$. Thus

(8) $$n = \sum_1^N (\beta_i - m_i) f_i , \text{ with } -m_i \leq \beta_i - m_i \leq k_i ,$$

and the identification $\alpha_i = \beta_i - m_i$ for $i = 1, 2, \ldots, N$ shows that $f_i(\infty)$ is quasi-complete.

Conversely, assume $f_i(\infty)$ is quasi-complete. Then by Definition 1, the inequality

$$0 < n < 1 + \sum_1^N k_i f_i \quad \text{implies}$$

$$n = \sum_1^N \alpha_i f_i \quad \text{with } -m_i \leq \alpha_i \leq k_i ,$$

and we wish to show that (6) is satisfied.

For a proof by contradiction, assume (6) does not hold; then, there exists an integer r greater than zero such that

(9) $$f_{r+1} > 1 + \sum_1^r (k_i + m_i) f_i .$$

Hence,

$$0 < f_{r+1} - \sum_{1}^{r} m_i f_i - 1 < f_{r+1} < 1 + \sum_{1}^{r+1} k_i f_i ,$$

and by the quasi-completeness of $f_i(\infty)$,

(10) $$f_{r+1} - \sum_{1}^{r} m_i f_i - 1 = \sum_{1}^{r+1} \alpha_i f_i \text{ with } -m_i \le \alpha_i \le k_i .$$

Now, in the representation (10), $\alpha_{r+1} > 0$, for, if not, then

$$f_{r+1} = \alpha_{r+1} f_{r+1} + 1 + \sum_{1}^{r} (\alpha_i + m_i) f_i$$

$$\le 1 + \sum_{1}^{r} (\alpha_i + m_i) f_i \le 1 + \sum_{1}^{r} (k_i + m_i) f_i$$

in violation of assumption (9). Thus, from (10)

(11) $$- \sum_{1}^{r} (\alpha_i + m_i) f_i - 1 = (\alpha_{r+1} - 1) f_{r+1} \ge 0.$$

But the left hand side of (11) is clearly ≤ -1, giving the desired contradiction. We conclude that (6) must be satisfied for all values of $p \ge 1$.

For nondecreasing sequences, quasi-completeness can be rephrased in terms of semi-completeness according to the following Corollary:

COROLLARY 1: A nondecreasing sequence of positive integers $f_i(\infty)$ is quasi-complete with respect to the sequences

$$k_i(\infty) \text{ and } m_i(\infty)$$

iff $f_i(\infty)$ is semi-complete with respect to the sequence

$$\{k_i + m_i\}$$

PROOF: From [2] and Theorem 1, the necessary and sufficient condition for both statements is inequality (6).

COROLLARY 2: If a nondecreasing sequence of positive integers $f_i(\infty)$ with $f_1 = 1$ is complete, then it is also quasi-complete with respect to arbitrary sequences, $k_i(\infty)$ and $m_i(\infty)$.

PROOF: By Theorem 1 of [1],

$$f_{p+1} \leq 1 + \sum_1^p f_i \leq 1 + \sum_1^p (k_i + m_i) f_i \quad \text{for } p = 1, 2, \ldots$$

for arbitrary sequences $k_i(\infty)$ and $m_i(\infty)$, since $k_i \geq 1$ and $m_i \geq 0$ for all $i \geq 1$.

COROLLARY 3: If a nondecreasing sequence of positive integers $f_i(\infty)$ with $f_1 = 1$ is semi-complete with respect to the sequence of positive integers $k_i(\infty)$, then it is also quasi-complete with respect to the same sequence $k_i(\infty)$ and any sequence $m_i(\infty)$ of nonnegative integers.

PROOF: From Theorem 1 of [2],

$$f_{p+1} \leq 1 + \sum_1^p k_i f_i \leq 1 + \sum_1^p (k_i + m_i) f_i ,$$

and the Corollary is immediate from the characterization of quasi-completeness. Alternatively, Corollary 1 implies the result since semi-completeness with respect to $k_i(\infty)$ implies semi-completeness with respect to $\{k_i + m_i\}$.

Before discussing uniqueness, we note that, for given sequences $k_i(\infty)$ and $m_i(\infty)$, quasi-completeness is a sufficient condition for every positive integer to possess a representation in the form of equation (1). However, if the m_i are not all zero, then quasi-completeness is not necessary for such representations even in the case of a nondecreasing sequence $f_i(\infty)$. For, let $k_i = m_i = 1$ for all $i \geq 1$, and consider the particular sequence

$$f_i(\infty) = \{1, 10, 100, 101, 102, 103, 104, 105, \ldots\}$$

Then the inequality

$$f_{p+1} \leq 1 + 2 \sum_1^p f_i \quad \text{is not satisfied for } p=1, 2\ldots;$$

nevertheless, any positive integer n has a representation in the prescribed form:

$$n = (102-101) + (104-103) + \ldots + [(100+2n)-(100+2n-1)]$$

Clearly, the same situation obtains for any sequence $f_i(\infty)$ which contains all consecutive integers after some fixed index $n = n_0$, where n_0 may be arbitrarily large. This shows that in order to obtain a necessary condition which holds for all members of the sequence, some additional constraint must be introduced. The one chosen in the above theorem requires that whenever

$$n < 1 + \sum_1^N k_i f_i \text{, the representation for n can be}$$

accomplished in terms of the first N members of the sequence. Thus, for a quasi-complete sequence, every positive integer which is

$$\leq \sum_1^N k_i f_i$$

can be represented in the proper form using only the terms $f_1, f_2, f_3, \ldots\ldots f_N$. But the largest number that can be represented in the proper form using only these terms is

$$\sum_1^N k_i f_i$$

so that, in this sense, the condition is the best possible.

In order to discuss uniqueness of the representation, we introduce, for given sequences $k_i(\infty)$ and $m_i(\infty)$, the particular sequence of positive integers $\varphi_i(\infty)$, defined by

$$\text{(12)} \qquad \begin{aligned} \varphi_1 &= 1 \\ \varphi_{p+1} &= 1 + \sum_1^p (k_i + m_i)\,\varphi_i, \quad \text{for } p \geq 1. \end{aligned}$$

It is straightforward to show that the terms of this sequence may also be written in the equivalent form:

$$\text{(13)} \qquad \begin{aligned} \varphi_1 &= 1 \\ \varphi_{p+1} &= \prod_{i=1}^{p} (1 + k_i + m_i), \quad \text{for } p \geq 1. \end{aligned}$$

DEFINITION 4: For given sequences $k_i(\infty)$ and $m_i(\infty)$, a sequence of positive integers $f_i(\infty)$ will be said to possess the <u>uniqueness property</u> iff for any $N > 0$, the equation

$$\text{(14)} \qquad \sum_1^N \alpha_i f_i = \sum_1^N \beta_i f_i, \text{ with } -m_i \leq \alpha_i \leq k_i$$

$$\text{and } -m_i \leq \beta_i \leq k_i, \qquad i = 1, 2, \ldots, N$$

implies $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, N$. (In other words, every integer, positive or negative, which possesses a representation in the required form has only one such representation in that form.)

THEOREM 2: Let $k_i(\infty)$ and $m_i(\infty)$ be given and let $f_i(\infty)$ be a quasi-complete sequence of positive integers with $f_1 = 1$.

Then $f_i(\infty)$ possesses the uniqueness property iff

$$\text{(15)} \qquad f_i = \varphi_i \quad \text{for } i = 1, 2, 3, \ldots.$$

PROOF: Assume $f_i(\infty)$ possesses the uniqueness property and that $f_i = \varphi_i$ does not hold for all $i \geq 1$. Then there exists a least integer $N > 0$ such that $F_N \neq \varphi_N$. (Note that $N > 1$). From the quasi-completeness, we have

$$0 < f_N \leq 1 + \sum_1^{N-1} (k_i + m_i) f_i = 1 + \sum_1^{N-1} (k_i + m_i) \varphi_i = \varphi_N$$

Since $f_N \neq \varphi_N$, we must have $f_N < \varphi_N$ and

$$0 < f_N < 1 + \sum_1^{N-1} (k_i + m_i) \varphi_i .$$

By Lemma 1, f_N can be written in the form

(16) $f_N = \sum_1^{N-1} \gamma_i \varphi_i = \sum_1^{N-1} \gamma_i f_i$ where each γ_i is a nonnegative integer satisfying $0 \leq \gamma_i \leq k_i + m_i$.

Hence,

(17) $$f_N - \sum_1^{N-1} m_i f_i = \sum_1^{N-1} (\gamma_i - m_i) f_i ,$$ where

$$-m_i \leq \gamma_i - m_i \leq k_i .$$

Applying the uniqueness property to (17), we find that $\gamma_i - m_i = -m_i$ or $\gamma_i = 0$ for $i = 1, 2, \ldots\ldots, N-1$. Thus, from (16), $f_N = 0$, a contradiction, and we conclude that $f_i = \varphi_i$ for all $i \geq 1$.

For the converse, we must show that $\varphi_i(\infty)$ possesses the uniqueness property. The proof is by contradiction. If $\varphi_i(\infty)$ does not possess the uniqueness property, then there is a <u>least</u> integer $N > 0$ such that $\alpha_i(N)$ and $\beta_i(N)$ exist having the property

(18) $$\sum_1^{N} \alpha_i \varphi_i = \sum_1^{N} \beta_i \varphi_i \quad \text{with } -m_i \leq \alpha_i \leq k_i$$

and $-m_i \le \beta_i \le k_i$, $(i = 1, 2, \ldots, N)$

and $\sum_1^N | \alpha_i - \beta_i | \neq 0$.

Clearly, $N > 1$, since $\alpha_1 \varphi_1 = \beta_1 \varphi_1$ implies $\alpha_1 = \beta_1$. Moreover, we assert that $\alpha_N \neq \beta_N$ in (18). For if $\alpha_N = \beta_N$, then

$$\sum_1^{N-1} \alpha_i \varphi_i = \sum_1^{N-1} \beta_i \varphi_i, \text{ with } -m_i \le \alpha_i \le k_i,$$

$$-m_i \le \beta_i \le k_i, \quad (i = 1, 2, \ldots, N-1)$$

and $\sum_1^{N-1} | \alpha_i - \beta_i | \neq 0$. But this contradicts our choice of N as the smallest upper limit affording two distinct representations. Hence $\alpha_N \neq \beta_N$.

From (18)

$$(19) \qquad (\beta_N - \alpha_N)\varphi_N = \sum_1^{N-1} (\alpha_i - \beta_i)\varphi_i ,$$

and therefore,

$$(20) \qquad \varphi_N \le | \beta_N - \alpha_N | \varphi_N \le \sum_1^{N-1} | \alpha_i - \beta_i | \varphi_i$$

$$\le \sum_1^{N-1} (k_i + m_i)\varphi_i = \varphi_N - 1,$$

giving a contradiction.

EXAMPLES: (a) As our first example, we consider the sequence of pth powers of the Fibonacci numbers, where $p \ge 2$. It is known [1] that the sequence $F_i(\infty)$, which is defined as $\{1, 1, 2, 3, 5, \ldots\}$, is complete; therefore, every positive integer n has a representation in the form

$$n = \sum_{1}^{M} c_i F_i \text{ , where each } c_i \text{ is either 0 or 1.}$$

To generalize this result, we leave it to the reader to verify the following inequality:

$$(21) \qquad F^p_{n+1} \leq 1 + 2^{p-1} \sum_{1}^{n} F^p_i \text{ , where p is a fixed integer}$$

greater than or equal to 1. From (21), it is clear that, for sequences $k_i(\infty)$ and $m_i(\infty)$ defined by $k_i = m_i$,

$$m_i = 2^{p-2}, \text{ for all } i \geq 1, \text{ the sequence}$$

$F^p_i(\infty)$ is quasi-complete. Thus every positive integer n has a representation in the form

$$(22) \qquad n = \sum_{1}^{\infty} \alpha_i F^p_i \text{ , where } \alpha_i \text{ is an integer}$$

$$\text{satisfying } |\alpha_i| \leq 2^{p-2}, \text{ for } i \geq 1.$$

Moreover, from Theorem 1, if N is chosen so that

$$0 < n < \sum_{1}^{N} 2^{p-2} F^p_i \text{ ,}$$

then n has at least one representation in the form (22) which uses only the terms

$$F^p_1 , F^p_2 , \ldots\ldots\ldots , F^p_N .$$

In particular, $0 < n < 1 + \sum_{1}^{N} F^2_i$ implies

$$n = \sum_{1}^{N} \alpha_i F^2_i \text{ , where each } \alpha_i \text{ is}$$

either -1, 0, or +1.

(b) To illustrate Theorem 2, let $k_i(\infty)$ and $m_i(\infty)$ be defined by $k_i \equiv m_i \equiv 1$ for all $i \geq 1$. Then, if a given sequence $f_i(\infty)$ is quasi-complete with respect to $k_i(\infty)$ and $m_i(\infty)$, every positive integer n has a representation

$$n = \sum_1^\infty \alpha_i f_i, \text{ where each } \alpha_i \text{ is either } -1, 0, \text{or } +1$$

Next, define [compare (13)]

$$\varphi_1 = 1$$

(23) $$\varphi_{N+1} = \prod_1^n (1 + k_i + m_i) = \prod_1^n 3 = 3^n, \text{ for } n \geq 1.$$

Then $\varphi_n = 3^{n-1}$ for all $n \geq 1$, and since $\varphi_{n+1} = 1 + 2\sum_1^n \varphi_i$, the sequence $\varphi_i(\infty)$ is quasi-complete with respect to the unity sequences, $k_i(\infty)$ and $m_i(\infty)$. Moreover, according to Theorem 2, representations are unique in the sense that if

$$\sum_1^M \alpha_i \varphi_i = \sum_1^M \beta_i \varphi_i \quad \text{with } |\alpha_i| \leq 1 \text{ and } |\beta_i| \leq 1 \text{ for}$$

$i \geq 1$, then $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, M$.

Combining Theorems 1 and 2, we have that every integer n satisfying

$$0 < n < 1 + \sum_1^N 3^{i-1} = \frac{3^N + 1}{2}$$

(namely, the integers $1, 2, 3, \ldots, \frac{3^N - 1}{2}$) has a <u>unique</u> representation in the form

$$n = \sum_1^N \alpha_i 3^{i-1} \quad \text{with each } \alpha_i = -1, 0, \text{or } +1$$

The reader may note that this result provides a solution to Bachet's weighing problem [3] . It is also left to the reader to interpret the quasi-complete sequence $\varphi_i(\infty)$ of (13) as the solution of a dual-pan weighing problem with the constraint that at most k_i weights of magnitude φ_i can be used in the right pan, at most m_i weights of magnitude φ_i can be used in the left pan, and every integral number of pounds less than or equal to

$$\sum_{1}^{N} k_i \varphi_i$$

must be weighable using only the weights, $\varphi_1, \varphi_2, \ldots \varphi_N$.

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HAVE YOU SEEN?

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