GENERATING FUNCTIONS FOR PRODUCTS OF POWERS OF FIBONACCI NUMBERS*

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1. INTRODUCTION

We may define the Fibonacci numbers, F_n, by $F_0 = 0, F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. A well-known generating function for these numbers is

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n . \tag{1.1}$$

Intimately associated with the numbers of Fibonacci are the numbers of Lucas, L_n, which we may define by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$. The numbers F_n and L_n may be considered as special cases of general functions first studied in great detail by Lucas [8], though as Bell [1] has observed many expansions for the Lucas functions appeared in papers of Cauchy and others prior to Lucas. Dickson [4] devotes all of one chapter (17) to recurring series and more particularly Lucas functions. Here one may find further references to the many papers on the subject which have appeared since Leonardo Pisano, or Fibonacci, first introduced the famous numbers in 1202. It would be difficult to estimate how many papers related to Fibonacci numbers have appeared since Dickson's monumental History was written, however it may be of interest to point out that a project has been initiated under the direction of Professor Vern Hoggatt, San Jose State College, San Jose, California, to collect formulas, maintain a bibliography and coordinate work on Fibonacci numbers. As part of the writer's activity with this Fibonacci Bibliographical Project the subject of generating functions for powers of the Fibonacci numbers has come in for some study, and the object of this present paper is to develop some very general generating functions for the Lucas functions.

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Riordan [10] has recently made a very interesting study of arithmetic properties of certain classes of coefficients which arose in his analysis of the generating function defined by the p-th powers of Fibonacci numbers.

$$f_p(x) = \sum_{n=0}^{\infty} f_n^p x^n , \tag{1.2}$$

where $f_n = F_{n+1}$. Golomb [5] had found essentially that for squares of Fibonacci numbers we have

$$(1 - 2x - 2x^2 + x^3) f_2(x) = 1 - x$$

and it was this which led Riordan to seek the general form of $f_p(x)$.

However, there are other simple generating functions for the numbers of Fibonacci. First of all, let us observe that we may define the Fibonacci and Lucas numbers by

$$F_n = \frac{a^n - b^n}{a - b} , \quad L_n = a^n + b^n \tag{1.3}$$

where

$$a = \frac{1}{2}(1 + \sqrt{5}) , \qquad b = \frac{1}{2}(1 - \sqrt{5}) . \tag{1.4}$$

The very general functions studied by Lucas, and generalized by Bell [1, 2], are essentially the F_n and L_n defined by (1.3) with a, b being the roots of the quadratic equation $x^2 = Px - Q$ so that $a + b = P$ and $ab = Q$. In view of this formulation it is easy to show that we also have the generating function

$$\frac{e^{ax} - e^{bx}}{a - b} = \sum_{n=0}^{\infty} \frac{x^n}{n!} F_n . \tag{1.5}$$

For many purposes this expansion is easier to consider than (1.1), and one is naturally led to ask what form of generating function holds if we put p-th powers of the

Fibonacci numbers F_n in (1.5). Similar questions arise for L_n. We shall also consider negative powers of. F_n, L_n, and suggest an analogy with the polynomials of Bernoulli and Euler.

2. GENERATING FUNCTIONS FOR LUCAS FUNCTIONS

Suppose we are given for any initial generating function, $F(x)$, say

$$F(x) = \sum_{n=0}^{\infty} A_n x^n \tag{2.1}$$

with no particular restrictions on A_n. It follows at once from this that

$$F(ax) + F(bx) = \sum_{n=0}^{\infty} A_n x^n(a^n + b^n) = \sum_{n=0}^{\infty} A_n x^n L_n \tag{2.2}$$

and

$$\frac{F(ax) - F(bx)}{a - b} = \sum_{n=0}^{\infty} A_n x^n F_n \quad . \tag{2.3}$$

This incidentally is rather like the method used by Riordan [10] to begin his study of recurrence relations for the generation function (1.2), except that we could study (1.5) as well as (1.1) in the general expansion of (2.3).

Now of course we may iterate upon the formulas (2.2) and (2.3) by making successive substitutions, replacing x by ax, or bx, and adding or subtracting, and by this iteration build up generating functions involving F_n^p and L_n^p. Thus we have from (2.2)

$$F(a^2x) + F(abx) = \Sigma A_n x^n a^n L_n \quad ,$$

$$F(abx) + F(b^2x) = \Sigma A_n x^n b^n L_n \quad ,$$

so that

$$F(a^2x) + 2F(abx) + F(b^2x) = \sum_{n=0}^{\infty} A_n x^n L_n^2 , \tag{2.4}$$

and in similar fashion

$$\frac{F(a^2x) - 2F(abx) + F(b^2x)}{(a - b)^2} = \sum_{n=0}^{\infty} A_n x^n F_n^2 . \tag{2.5}$$

Clearly we may proceed inductively to obtain a general result. We find the general relations

$$\sum_{k=0}^{p} \binom{p}{k} F(a^{p-k} b^k x) = \sum_{n=0}^{\infty} A_n x^n L_n^p , \tag{2.6}$$

and

$$(a - b)^{-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} F(a^{p-k} b^k x) = \sum_{n=0}^{\infty} A_n x^n F_n^p . \tag{2.7}$$

In fact we may readily combine the relations to obtain

$$\begin{aligned} &(a - b)^{-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \cdot \sum_{j=0}^{q} \binom{q}{j} \\ &F(a^{p+q-k-j} b^{k+j} x) = \sum_{n=0}^{\infty} A_n x^n F_n^p L_n^q , \end{aligned} \tag{2.8}$$

for any non-negative integers p, q. Thus in principle we may set down generating functions for products of powers of the Fibonacci and Lucas numbers, though the result may not usually be in the simplest form.

We obtain (1.1) when $A_n = 1$ identically; (1.5) when $A_n = 1/n!$ identically. The expansion analogous to (1.5) for L_n is

$$(2.9) \qquad e^{ax} + e^{bx} = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n .$$

Proceding in the same manner as above, it is clear that we also have

$$(2.10) \qquad F(a^m x) + F(b^m x) = \sum_{n=0}^{\infty} A_n x^n L_{mn} ,$$

and

$$(2.11) \qquad \frac{F(a^m x) - F(b^m x)}{a - b} = \sum_{n=0}^{\infty} A_n x^n F_{mn} ,$$

which include other well-known generating functions. Consequently we have

$$(2.12) \qquad \sum_{k=0}^{p} \binom{p}{k} F(a^{pm-km} b^{km} x) = \sum_{n=0}^{\infty} A_n x^n L_{mn}^p ,$$

with a corresponding result for F_n.

3. RECIPROCALS OF FIBONACCI NUMBERS

Landau [7] showed that a certain series of reciprocals of Fibonacci numbers could be expressed in terms of a Lambert series. In fact he showed that if we write

$$(3.1) \qquad L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} ,$$

then we have

$$(3.2) \qquad \sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5} \left[L\left(\frac{3 - \sqrt{5}}{2}\right) - L\left(\frac{7 - 3\sqrt{5}}{2}\right) \right] .$$

The device used to obtain this is to expand $(a - b)/(a^n - b^n) = (a - b)/a^n \cdot 1/(1 - z)$, where $z = (b/a)^n$, by a power series, and then invert the order of summation in the series, this being justifiable. Landau's result is noted in Bromwich [3, p. 194, example 32], in Knopp [6, p. 279, ex. 144; p. 468, ex. 9], and in Dickson [4, p. 404]. Dickson also notes that $\sum_1^\infty 1/F_n$ was put in finite form by A. Arista.

Let us now define in general

$$R(x) = \sum_{n=1}^{\infty} A_n \frac{x^n}{F_n} . \tag{3.3}$$

Then by the same technique we have used earlier to obtain (2.2) and (2.3) we see at once that $R(x)$ satisfies a functional equation

$$R(ax) - R(bx) = (a - b) \sum_{n=1}^{\infty} A_n x^n . \tag{3.4}$$

Thus if we have

$$R(x) = \sum_{n=1}^{\infty} \frac{x^n}{F_n} \quad \text{then} \quad R(ax) - R(bx) = (a - b)\frac{x}{1 - x} \tag{3.5}$$

$$= \frac{x}{1 - x}\sqrt{5} \quad \text{for ordinary Fibonacci numbers.}$$

For $R(x)$ as defined by (3.3) we also easily verify that

$$R(a^2x) - R(b^2x) = (a - b) \sum_{n=1}^{\infty} A_n x^n L_n . \tag{3.6}$$

To use this when $A_n = 1$ we need to note the generating function for L_n which is a companion to (1.1). Let us obtain this from (2.2). Inasmuch as we have

$$a + b = 1, \quad a - b = \sqrt{5}, \quad \text{and} \quad ab = -1, \tag{3.7}$$

we find

$$\begin{aligned} F(ax) + F(bx) &= \frac{1}{1 - ax} + \frac{1}{1 - bx} \\ &= \frac{2 - (a + b)x}{1 - (a + b)x + abx} = \frac{2 - x}{1 - x - x^2} \end{aligned}$$

and so for the ordinary Lucas numbers we find

$$\frac{2 - x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_n x^n . \tag{3.8}$$

Sometimes this is stated in the equivalent form (since $L_0 = 2$)

$$\frac{x(2x + 1)}{1 - x - x^2} = \sum_{n=1}^{\infty} L_n x^n . \tag{3.9}$$

Thus if we have

$$R(x) = \sum_{n=1}^{\infty} \frac{x^n}{F_n} \quad \text{then} \quad R(a^2x) - R(b^2x) = (a - b) \cdot \frac{x(2x + 1)}{1 - x - x^2} . \tag{3.10}$$

4. BILINEAR GENERATING FUNCTIONS

We wish to turn next to some simple results for what are called bilinear generating functions for Fibonacci and Lucas numbers. To discuss this we first introduce what we shall call a general Turán operator defined by

$$Tf = T_x f(x) = f(x + u) f(x + v) - f(x) f(x + u + v) . \tag{4.1}$$

For the Fibonacci numbers it is a classic formula first discovered apparently by Tagiuri (Cf. Dickson [4, p. 404]) and later given as a problem in the American Mathematical Monthly (Problem E 1396) that

$$T_n F_n = F_{n+u} F_{n+v} - F_n F_{n+u+v} = (-1)^n F_u F_v . \tag{4.2}$$

We may determine a general bilinear generating function for the series $\Sigma A_n x^n F_{n+u} F_{n+v}$ if we can first determine a result for $\Sigma A_n x^n F_n F_{n+u+v}$. To do this let us consider $\Sigma A_n x^n F_n F_{n+j}$.

Again, let us set as in (2.1)

$$F(x) = \sum_{n=0}^{\infty} A_n x^n$$

so that

$$x^j \cdot F(x) = \sum_{n=0}^{\infty} A_n x^{n+j} .$$

Then we find

$$a^j x^j F(ax) - b^j x^j F(bx) = \sum_{n=0}^{\infty} A_n x^{n+j} (a^{n+j} - b^{n+j})$$

and hence ultimately

$$\frac{a^j F(ax) - b^j F(bx)}{a - b} = \sum_{n=0}^{\infty} A_n x^n F_{n+j} . \tag{4.3}$$

Next we introduce F_n by the same device and we find

$$\frac{a^j F(a^2 x) - b^j F(abx)}{a - b} - \frac{a^j F(abx) - b^j F(b^2 x)}{a - b} = \Sigma A_n x^n F_{n+j} (a^n - b^n) ,$$

and consequently we have

$$(4.4)\qquad \frac{a^j F(a^2x) - (a^j + b^j) F(abx) + b^j F(b^2x)}{(a-b)^2} = \sum_{n=0}^{\infty} A_n x^n F_n F_{n+j} .$$

Moreover, since we have the Turán expression (4.2), we have

$$(4.5)\qquad \sum_{n=0}^{\infty} A_n x^n F_{n+u} F_{n+v} = \sum_{n=0}^{\infty} A_n x^n F_n F_{n+u+v} + F_u F_v \cdot F(-x) ,$$

where the relation (4.4) is used to simplify the right-hand side. In principle at least we have a way to write down explicit bilinear generating functions provided merely that we have $F(x) = \Sigma A_n x^n$ given.

It should be remarked that since we have made no special assumptions about the coefficients A_n in any of the work so far, we could apply our work to finite series just as well by supposing that $A_n = 0$ identically for $n \geq$ some value n_0.

As far as relation (4.4) is concerned, there is an alternative method. Call the series $M_j(x)$, i.e., let

$$M_j(x) = \sum_{n=0}^{\infty} A_n x^n F_n F_{n+j} .$$

If it is possible to evaluate $M_0(x) = \Sigma A_n x^n F_n^2$, then one may note that $M_{j+1}(x) = M_j(x) + M_{j-1}(x)$ and so a simple formula could be written down giving M_j.

5. BERNOULLI AND EULER POLYNOMIALS

Returning to the problems presented by reciprocals of Fibonacci and Lucas numbers, it would appear to be of value to introduce some new polynomials based upon the polynomials of Bernoulli and Euler. Using a standard notation [9] we define Euler and Bernoulli polynomials by

$$(5.1)\qquad \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} = \frac{2e^{tx}}{e^t + 1} ,$$

and

$$\sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \frac{t e^{tx}}{e^t - 1} . \tag{5.2}$$

Now we have for general Lucas functions

$$\frac{1}{L_t} = \frac{1}{a^t + b^t} = \frac{1}{a^t} \frac{1}{C^t + 1} \quad \text{with } C = b/a ,$$

so that

$$\frac{2 a^t C^{tx}}{L_t} = \frac{2 C^{tx}}{C^t + 1} \quad \text{where } C = b/a . \tag{5.3}$$

Similarly we find that when $F_t = (a^t - b^t)/(a - b)$ we have

$$\frac{t a^t C^{tx}}{(b - a)F_t} = \frac{t C^{tx}}{C^t - 1} \quad \text{where } C = b/a . \tag{5.4}$$

The similarity of (5.3) with (5.1) and (5.4) with (5.2) motivates what follows.

We define generalized Bernoulli and Euler polynomials by

$$\frac{t C^{tx}}{C^t - 1} = \sum_{k=0}^{\infty} B_k(x, C) \frac{t^k}{k!} , \tag{5.5}$$

and

$$\frac{2 C^{tx}}{C^t + 1} = \sum_{k=0}^{\infty} E_k(x, C) \frac{t^k}{k!} . \tag{5.6}$$

Now in fact

$$(5.7) \qquad \frac{t\,C^{tx}}{C^t - 1} = \frac{1}{\log C} \cdot \frac{t \log C \cdot e^{x(t \log C)}}{e^{t \log C} - 1} = \frac{1}{\log C} \cdot \frac{z\,e^{xz}}{e^z - 1}$$

$$= \frac{1}{\log C} \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} (\log C)^{k-1}$$

so that

$$(5.8) \qquad B_k(x, C) = (\log C)^{k-1} \cdot B_k(x) \ .$$

Similarly one easily finds that

$$(5.9) \qquad E_k(x, C) = (\log C)^k \cdot E_k(x) \ .$$

Putting these observations together we ultimately have the expansions

$$(5.10) \qquad \frac{1}{L_t} = \frac{1}{2\,a^t\,(b/a)^{tx}} \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} (\log b/a)^k \ ,$$

and

$$(5.11) \qquad \frac{1}{F_t} = \frac{b - a}{t\,a^t (b/a)^{tx}} \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} (\log b/a)^{k-1} \ .$$

Thus we also have an amusing analogy between four names:

$$(5.12) \qquad \frac{\text{BERNOULLI}}{\text{FIBONACCI}} = \frac{\text{EULER}}{\text{LUCAS}}$$

We may extend the analogy by considering the more general Bernoulli and Euler polynomials of higher order as discussed in [9] and find expansions for the reciprocals of powers of the numbers of Fibonacci and Lucas.

We have

$$\sum_{k=0}^{\infty} B_k^{(n)}(x)\,\frac{t^k}{k!} = \frac{t^n e^{xt}}{(e^t - 1)^n}, \tag{5.13}$$

and

$$\sum_{k=0}^{\infty} E_k^{(n)}(x)\,\frac{t^k}{k!} = \frac{2^n e^{xt}}{(e^t + 1)^n}, \tag{5.14}$$

and we ultimately find as before the reciprocal expansions (with $C = b/a$)

$$\frac{t^n C^{xt}}{(C^t - 1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(x)\,\frac{t^k (\log C)^{k-n}}{k!} = \frac{t^n a^{nt} C^{tx}}{(b - a)^n F_t^n}, \tag{5.15}$$

and

$$\frac{2^n C^{xt}}{(C^t + 1)^n} = \sum_{k=0}^{\infty} E_k^{(n)}(x)\,\frac{t^k (\log C)^k}{k!} = \frac{2^n a^{nt} C^{tx}}{L_t^n}. \tag{5.16}$$

Consequently we have

$$\sum_{t=m}^{\infty} A_t \frac{z^t}{F_t^n} = (b - a)^n \sum_{k=0}^{\infty} B_k^{(n)}(x)\,\frac{(\log C)^{k-n}}{k!} \sum_{t=m}^{\infty} A_t t^{k-n}\left(\frac{z}{a^n C^x}\right)^t, \tag{5.17}$$

$$m \geq 1$$

and

$$\sum_{t=m}^{\infty} A_t \frac{z^t}{L_t^n} = 2^{-n} \sum_{k=0}^{\infty} E_k^{(n)}(x) \frac{(\log C)^k}{k!} \sum_{t=m}^{\infty} A_t t^k \left(\frac{z}{a^n C^x}\right)^t , \tag{5.18}$$

$$m \geq 0 .$$

In these, log C will be real provided, e.g. that both a and b are positive, or both negative. In case C is negative we may take principal values for the log C.

6. SOME MISCELLANEOUS FIBONACCI FORMULAE

We shall conclude our remarks here by deriving a few miscellaneous relations.

In relation (2.1) let $A_n = \binom{z}{n}$, z being any real number. Then we find by (2.2)

$$\sum_{n=0}^{\infty} \binom{z}{n} x^n L_n = (1 + ax)^z + (1 + bx)^z , \tag{6.1}$$

and by (2.3)

$$\sum_{n=0}^{\infty} \binom{z}{n} x^n F_n = \frac{(1 + ax)^z - (1 + bx)^z}{a - b} . \tag{6.2}$$

Let us examine one special instance. Let $z = r$ be any non-negative integer, and take the ordinary Fibonacci-Lucas numbers when $a = \frac{1}{2}(1 + \sqrt{5})$, $b = \frac{1}{2}(1 - \sqrt{5})$. Then we find $1 + a = a^2$, $1 + b = b^2$, whence (6.1) and (6.2) become

$$\sum_{n=0}^{r} \binom{r}{n} L_n = (1 + a)^r + (1 + b)^r = a^{2r} + b^{2r} = L_{2r} , \tag{6.3}$$

and

$$\sum_{n=0}^{r} \binom{r}{n} F_n = \frac{a^{2r} - b^{2r}}{a - b} = F_{2r} . \tag{6.4}$$

It is clear that by using the general relations previously developed here, we could go on to derive many interesting Fibonacci-Lucas number relations. As another example using the same value for A_n, we find from (2.10) that

$$\sum_{n=0}^{r} \binom{r}{n} x^n L_{mn} = (1 + a^m x)^r + (1 + b^m x)^r \quad (6.5)$$

In this, let $x = -1$, $m = 2$, and a and b as above. Then we find

$$\sum_{n=0}^{r} \binom{r}{n} (-1)^n L_{2n} = (-1)^r L_r \quad (6.6)$$

Also we have

$$\sum_{n=0}^{r} \binom{r}{n} (-1)^n F_{2n} = (-1)^r F_r \quad (6.7)$$

By (4.3) we have

$$\sum_{n=0}^{\infty} A_n x^n F_{n+j} = \frac{a^j F(ax) - b^j F(bx)}{a - b} ,$$

and similarly we have

$$\sum_{n=0}^{\infty} A_n x^n L_{n+j} = a^j F(ax) + b^j F(bx) , \quad (6.8)$$

where as before in (2.1) we have $F(x) = \sum_{n=0}^{\infty} A_n x^n$.

Let $A_n = \binom{r}{n}$ and take $x = 1$. We then have for the ordinary Fibonacci-Lucas numbers

$$(6.9)\qquad \sum_{n=0}^{r} \binom{r}{n} F_{n+j} = F_{2r+j} \quad ,$$

and

$$(6.10)\qquad \sum_{n=0}^{r} \binom{r}{n} L_{n+j} = L_{2r+j} \quad ,$$

which are well-known recursions.

A very elegant symmetrical relation may be gotten from (2.6). In that relation we choose $A_n = \binom{r}{n}$ and set $x = a^{-p}$. The reader may easily verify that the formula then becomes, since $F(a^{p-k}b^k x) = (1 + b^k a^{-k})^r = L_k^r a^{-kr}$,

$$(6.11)\qquad \sum_{n=0}^{r} \binom{r}{n} \frac{L_n^p}{a^{pn}} = \sum_{n=0}^{p} \binom{p}{n} \frac{L_n^r}{a^{rn}} \quad .$$

Similarly in (2.7) if we set $A_n = \binom{r}{n}$ and take $x = -a^{-p}$ we find easily that

$$(6.12)\qquad (a - b)^p \sum_{n=0}^{r} (-1)^n \binom{r}{n} \frac{F_n^p}{a^{pn}} = (a - b)^r \sum_{n=0}^{p} (-1)^n \binom{p}{n} \frac{F_n^r}{a^{rn}} \quad .$$

And similarly we find from (2.8) that (here we take $x = a^{-p-q}$)

$$(6.13)\qquad \sum_{n=0}^{r} \binom{r}{n} a^{-n(p+q)} F_n^p L_n^q = (a - b)^{-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{q} \binom{q}{j} \frac{L_{k+j}^r}{a^{r(k+j)}} \quad .$$

With other choices of x we could give similar results. In fact with $x = -a^{-p-q}$ we have

$$(6.14)\qquad \sum_{n=0}^{r} (-1)^n \binom{r}{n} \frac{F_n^p L_n^q}{a^{n(p+q)}} = (a - b)^{r-p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{q} \binom{q}{j} \frac{F_{k+j}^r}{a^{r(k+j)}}$$

It may be of interest to note that Kelisky [12] developed some curious results involving Bernoulli, Euler, Fibonacci, and Lucas numbers. The relations he gives should be compared with those developed in the present paper. In particular, Kelisky has since written the present author that the unpublished proofs of the last collection of relations he found are somewhat similar to the methods of the present note.

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