ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY S.L. BASIN, SYLVANIA ELECTRONIC SYSTEMS, MT. VIEW, CALIF.

Send all communications regarding Elementary Problems and Solutions to S. L. Basin, 946 Rose Ave., Redwood City, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer must submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated. Solutions should be submitted within two months of the appearance of the problems.

B-17 *Proposed by Charles R. Wall, Ft. Worth, Texas*

If m is an integer, prove that

$$F_{n+4m+2} - F_n = L_{2m+1} F_{n+2m+1}$$

where F_p and L_p are the p^{th} Fibonacci and Lucas numbers, respectively.

B-18 *Proposed by J. L. Brown, Jr., Pennsylvania State University.*

Show that

$$F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1} \frac{\pi}{5} \sin^k \frac{\pi}{10}, \quad \text{for } n \geq 0 .$$

B-19 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}^2 F_{n+3}} + \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}^2 F_{n+3}} = \frac{1}{2} .$$

B-20 *Proposed by Louis G. Brökling, Redwood City, Calif.*

Generalize the well-known identities,

(i) $$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

(ii) $$L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3 .$$

B-21 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

If

$$u_n = \frac{1}{2}\left[(x+1)^{2^n} + (x-1)^{2^n}\right]$$

show that

$$u_{n+1} = u_n^2 + 2^{2n} u_0^2 u_1^2 \cdots u_{n-1}^2 .$$

B-22 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

Prove the Fibonacci identity

$$F_{2k}F_{2k'} = F_{k+k'}^2 - F_{k-k'}^2$$

and find the analogous Lucas identity.

B-23 *Proposed by S.L. Basin, Sylvania Electronic Systems, Mt. View, Calif.*

Prove the identities

(i) $$F_{n+1} = \prod_{i=1}^{n}\left(1 + \frac{F_{i-1}}{F_i}\right)$$

(ii) $$\frac{F_{n+1}}{F_n} = 1 + \sum_{i=1}^{n} \frac{(-1)^i}{F_i F_{i-1}}$$

(iii) $$\frac{1+\sqrt{5}}{2} = 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{F_i F_{i-1}}$$

SOLUTIONS TO PROBLEMS IN VOL. 1, FEBRUARY, 1963

B-1 Show that the sum of twenty consecutive Fibonacci numbers is divisible by F_{10}, i.e., $\sum_{i=1}^{20} F_{n+1} \equiv 0 \pmod{F_{10}}$, $n \geq 0$.

Solution by Marjorie R. Bicknell, San Jose State College, San Jose, Calif.

The proof is by induction. When $n = 0$

$$\sum_{i=1}^{20} F_i = F_{22} - 1 = F_{10}(F_{13} + F_{11}) .$$

Assume that the proposition is true for all $n \leq k$, i.e.,

$$\sum_{i=1}^{20} F_{k+i} \equiv 0 \pmod{F_{10}}$$

and

$$\sum_{i=1}^{20} F_{k-1+i} \equiv 0 \pmod{F_{10}}$$

Now addition of the congruences yields

$$\sum_{i=1}^{20} F_{k+i} + \sum_{i=1}^{20} F_{k-1+i} = \sum_{i=1}^{20} \left(F_{k+i} + F_{k-1+i}\right) = \sum_{i=1}^{20} F_{k+1+i} \equiv 0 \pmod{F_{10}}$$

Also solved by J. L. Brown, Jr., Dermott A. Breault, and the proposer.

B-2 Show that $u_{n+1} + u_{n+2} + \cdots + u_{n+10} = 11\,u_{n+7}$ holds for generalized Fibonacci numbers such that $u_{n+2} = u_{n+1} + u_n$, where $u_1 = p$ and $u_2 = q$.

Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania

It is easily shown, by induction, that u_n may be written in terms of the Fibonacci numbers, F_m, as $u_n = pF_{n-2} + qF_{n-1}$ for $n \geq 1$. Using this result, we have

$$\sum_{k=1}^{10} u_{n+k} = p\sum_{k=1}^{10} F_{n+k-2} + q\sum_{k=1}^{10} F_{n+k-1}$$

and

$$11\,u_{n+7} = 11\,(pF_{n+5} + qF_{n+6}) .$$

The result follows if

(1) $$\sum_{k=1}^{10} F_{n+k} = 11\,F_{n+7} \quad \text{for } n \geq 0 ;$$

however,

$$\sum_{k=1}^{10} F_{n+k} = \sum_{i=1}^{n+10} F_i - \sum_{i=1}^{n} F_i = (F_{n+12} - 1) + (F_{n+2} - 1)$$
$$= F_{n+12} - F_{n+2}, \quad n \geq 0 .$$

Equation (1) is therefore equivalent to

(2) $$F_{n+12} - F_{n+2} = 11\,F_{n+7} , \quad n \geq 0 .$$

By direct calculation, $F_{12} - F_2 = 11 F_7$ and $F_{13} - F_3 = 11 F_8$; now adding these identities we have, $F_{14} - F_4 = 11 F_9$. Proceeding in this fashion, (2) is verified by induction.

Also solved by Dermott A. Breault, Marjorie R. Bicknell and Edward Balizer.

B-3 Show that $F_{n+24} \equiv F_n \pmod 9$.

Solution by Marjorie R. Bicknell, San Jose State College, San Jose, Calif.

Proof is by mathematical induction. When $n = 0$,

$$F_{24} = F_{12}(F_{13} + F_{11}) = 144(F_{13} + F_{11}) \equiv F_0 \pmod 9)$$

Assuming that the proposition holds for all integers $n \leq k$,

$$F_{k-1+24} \equiv F_{k-1} \pmod 9$$

and

$$F_{k+24} \equiv F_k \pmod 9$$

Adding the congruences, we have

$$F_{k+1+24} \equiv F_{k+1} \pmod 9 ,$$

and the proof is complete by mathematical induction.

Solution by Dermott A. Breault, Sylvania ARL, Waltham, Mass.

Using the identity $F_{n+p+1} = F_{n+1}F_{p+1} + F_n F_p$ write

$$F_{24+n} = F_{23+n+1} = F_{24}F_{n+1} + F_{23}F_n .$$

Therefore,

$$F_{n+24} - F_n = F_{24}F_{n+1} + (F_{23} - 1)F_n , \text{ but}$$

$$F_{24} = 9(5153) \quad \text{and} \quad F_{23} - 1 = 9(3184) ,$$

hence

$$F_{n+24} - F_n \equiv 0 \pmod 9.$$

Also solved by J. L. Brown, Jr.

B-4 Show that

$$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$$

Generalize.

Solution by Joseph Erbacher, University of Santa Clara, Santa Clara, Calif., and J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.

Using the Binet formula,

$$F_{2kn} = \frac{(a^2)^{kn} - (b^2)^{kn}}{a - b}$$

where

$$a^2 = 1 + a, \quad b^2 = 1 + b, \quad a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2}$$

we have

$$F_{2kn} = \frac{(1 + a)^{kn} - (1 + b)^{kn}}{a - b} = \frac{1}{a - b}\left[\sum_{i=0}^{kn} \binom{n}{i} a^i - \sum_{i=0}^{kn} \binom{n}{i} b^i\right] = \sum_{i=0}^{kn} \binom{n}{i} \frac{a^i - b^i}{a - b}$$

$$= \sum_{i=0}^{kn} \binom{n}{i} F_i \quad .$$

Therefore,

$$F_{2kn} = \sum_{i=0}^{kn} \binom{n}{i} F_i$$

Also solved by the proposers.

B-5 Show that with order taken into account, in getting past an integral number N dollars, using only one-dollar and two-dollar bills, that the number of different ways is F_{N+1}.

Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.

Let α_N for $N \geq 1$ be the number of different ways of being paid N dollars in one and two dollar bills, taking order into account. Consider the case where $N \geq 2$. Since a one-dollar bill is received as the last bill if and only if N-1 dollars have been received previously and a two-dollar bill is received as the last bill if and only if N-2 dollars have been received previously, the two possibilities being mutually exclusive, we have $\alpha_N = \alpha_{N-1} + \alpha_{N-2}$ for $N \geq 2$. But $\alpha_1 = 1$, $\alpha_2 = 2$, therefore $\alpha_N = F_{N+1}$ for $N \geq 1$.

B-7 Show that
$$\frac{x(1-x)}{1-2x-2x^2+x^3} = \sum_{i=0}^{\infty} F_i^2 x^i .$$
Is the expansion valid at $x = 1\ 4$, i.e., does $\sum_{i=0}^{\infty} F_i^2\ 4^i = \frac{12}{25}$

Solution by Wm. E. Briggs, University of Colorado, Boulder, Colo.

Write $(1 - 2x - 2x^2 + x^3)^{-1} = \sum_{n=0}^{\infty} a_n x^n$ where

$$a_k = 2a_{k-1} + 2a_{k-2} - a_{k-3}, \quad k > 2, \text{ and } a_0 = 1, \ a_1 = 2, \ a = 6 .$$

Therefore,
$$\frac{x(1-x)}{1-2x-2x^2+x^3} = x + \sum_{n=2}^{\infty} (a_{n-1} - a_{n-2})\ x^n .$$

It follows that the coefficient of x^k is F_k^2 for $k = 1,2,3,4$; assume this is true for all $k \leq n$. From above, the coefficient of x^{n+1} is
$$a_n - a_{n-1} = 2(a_{n-1} - a_{n-2}) + 2(a_{n-2} - a_{n-3}) - (a_{n-3} - a_{n-4}).$$
Therefore, $a_n = a_{n-1} = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2$; however,
$$F_{n+1}^2 + F_{n-2}^2 = (F_n + F_{n-1})^2 + (F_n - F_{n-1})^2 = 2F_n^2 + 2F_{n-1}^2$$
so that the coefficient of x^{n+1} is F_{n-1}^2. The zero of $1 - 2x - 2x^2 + x^3$ with smallest modulus is $r = (1/2)(3 - \sqrt{5})$ which is the radius of convergence of the power series. Since $r > |1/4|$, the series converges for $x = 1/4$ to the value $12/25$.

Also solved by J. L. Brown, Jr.

B-8 Show that

(i) $F_{n+1} 2^n + F_n 2^{n+1} \equiv 1 \pmod 5$

(ii) $F_{n+1} 3^n + F_n 3^{n+1} \equiv 1 \pmod{11}$

(iii) $F_{n+1} 5^n + F_n 5^{n+1} \equiv 1 \pmod{29}$

Generalize.

Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.

The general result,
$$F_{n+1} p^n + F_n p^{n+1} \equiv 1 \pmod{p^2 + p - 1} ,$$
where p is a prime and $n \leq 0$ is proved by mathematical induction.

The proposition is clearly true for $n = 0$ and $n = 1$, with the usual definition $F_0 = 0$. Suppose the proposition is true for all $n \leq k$ where $k \geq 1$.

(Continued on p. 52)