We may use this recursion formula to substitute for the last row of the given determinant, D_n, and then apply standard row operations to get

$$D_n = \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ 2F_{n+1}^2 + 2F_n^2 - F_{n-1}^2 & 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2 & 2F_{n+3}^2 + 2F_{n+2}^2 - F_{n+1}^2 \end{vmatrix}$$

$$= \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ -F_{n-1}^2 & -F_n^2 & -F_{n+1}^2 \end{vmatrix} = -D_{n-1} .$$

It follows immediately by induction that $D_n = (-1)^{n-1} D_1$. Since $D_1 = 2$, $D_n = 2(-1)^{n-1} = 2(-1)^{n+1}$.

Also solved by Marjorie Bicknell and Dov Jarden.

Continued from p. 80, "Elementary Problems and Solutions"

Then

$$F_{k+2}p^{k+1} + F_{k+1}p^{k+2} = (F_{k+1} + F_k)p^{k+1} + (F_k + F_{k-1})p^{k+2}$$
$$= p(F_{k+1}p^k + F_kp^{k+1}) + p^2(F_kp^{k+1} + F_{k-1}p^k) .$$

But

$$p(F_{k+1}p^k + F_kp^{k+1}) + p^2(F_kp^{k-1} + F_{k-1}p^k) \equiv p + p^2 (\text{mod } p^2 + p - 1).$$

Since $F_{k+1}p^k + F_kp^{k+1}$ and $F_kp^{k-1} + F_{k-1}p^k$ are both congruent to 1 (mod $p^2 + p - 1$) by the induction hypothesis and $p + p^2 \equiv 1$ (mod $p^2 + p - 1$), the desired result follows by induction on n.

Also solved by Marjorie R. Bicknell and Donna J. Seaman.