FIBONACCI NUMBERS AND ZIGZAG HASSE DIAGRAMS*

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A Hasse diagram depicts the order relations in a partially ordered set. In this paper Haase diagrams will indicate the inclusion relations between members of a family of subsets of a given universe $U = \{e_1, \cdots, e_n\}$ of n elements. Each subset is represented by a vertex and an upward slanting segment is drawn from the vertex for a subset X to the vertex for a subset Y if X is contained in Y. [1]

In a previous paper the senior author described methods for finding the number f(n) of families $\{S_1, \cdots, S_r\}$ with each S_i a subset of U and with the inclusion relations among the S_i pictured by a given Hasse diagram. The formulas f(n) for all diagrams with $r = 2, 3$, or 4 were listed. The formulas for $r = 5$ have also been obtained and will be published subsequently.

We now single out a zigzag diagram for each $r \geq 2$, i.e., the diagrams

$$I, V, N, W, \cdots .$$

More precisely, we consider the problem of determining the number $a_r(n)$ of ordered r-tuples $(S_1, \cdots, S_r)$ of subsets S_i of U such that S_j is contained in S_k if and only if j is even and $k = j \pm 1$. Our previous results imply the formulas:

$$a_2(n) = 3^n - 2^n$$

$$a_3(n) = 5^n - 2 \cdot 4^n + 3^n$$

$$a_4(n) = 8^n - 3 \cdot 7^n + 3 \cdot 6^n - 5^n$$

$$a_5(n) = 13^n - 2 \cdot 12^n - 11^n + 5 \cdot 10^n - 4 \cdot 9^n + 8^n$$

$$a_6(n) = 21^n - 20^n - 2 \cdot 19^n - 18^n + 8 \cdot 17^n - 4 \cdot 16^n - 2 \cdot 15^n - 14^n + 3 \cdot 13^n - 12^n .$$

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Note that the leading term is the n^{th} power of the $(r + 2)$nd Fibonacci number. The object of this paper is to prove this for general r.

We begin by numbering the 2^r basic regions of the Venn diagram for r subsets S_i of U. Express a fixed integer k satisfying $0 \le k \le 2^r$ in binary form, i.e., let $k = c_1 + 2c_2 + 2^2c_3 + \cdots + 2^{r-1}c_r$ where each c_i is zero or one. For $i = 1, \cdots, r$ let W_i be S_i if $c_i = 1$ and let W_i be the complement of S_i in U if $c_i = 0$. Now let E_k be the intersection of $W_1, \cdots, W_r$. These E_k are the sets represented by the basic regions of the Venn diagram.

We next illustrate the process by finding $a_3(n)$. In this case the Hasse diagram is a V and we are concerned with ordered triples (S_1, S_2, S_3) such that S_2 is contained in S_1 and in S_3 and there are no other inclusion relations. The condition that S_2 is contained in S_1 forces E_2 and E_6 to be empty. The condition that S_2 is contained in S_3 forces E_3 (and E_2) to be empty. One then sees that there are no other inclusion relations if and only if both E_1 and E_4 are non-empty.

For a given triple (S_1, S_2, S_3) each of the n objects in the universe is in one and only one of the E_k. Excluding the empty E_2, E_3, and E_6, there are 5^n ways of distributing the n objects among the 5 remaining basic sets E_0, E_1, E_4, E_5, and E_7. We subtract the 4^n ways in which E_1 turns out to be empty (as well as E_2, E_3, and E_6) and also subtract the $4^n - 3^n$ ways in which E_4, but not E_1, is empty. The remaining $a_3(n) = 5^n - 4^n - (4^n - 3^n)$ ways of distributing the elements of U are all those that meet the conditions associated with the Hasse diagram V.

For a general r the inclusion relations of the zigzag diagram force $g(r)$ of the 2^r basic sets E_k to be empty. The technique illustrated above can be used to show that these are the E_k such that the r-tuple $(c_1, \cdots, c_r)$ of binary coefficients for k has an even-subscripted $c_i = 1$ with an adjacent $c_{i \pm 1} = 0$. The remaining r-tuples will be called <u>allowable</u>; there are $h(r) = 2^r - g(r)$ such r-tuples. We wish to show that $h(r)$ is the Fibonacci number F_{r+2}. It will then be clear that the leading term in $a_r(n)$ is $(F_{r+2})^n$ and that the other terms result from subtracting numbers of ways of distributing the elements of U among fewer E_k than the allowable ones.

For $r = 3$ the allowable triples are

$$(0, 0, 0), \quad (1, 0, 0), \quad (0, 0, 1), \quad (1, 0, 1), \quad (1, 1, 1), \tag{1}$$

i.e., those for E_0, E_1, E_4, E_5, and E_7. The allowable quadruples for $r = 4$ can be made by attaching a zero in the fourth place to the 2 triples in (1) that have a zero in the third place and by attaching either a zero or a one in the fourth place to each of the remaining 3 triples in (1). There are thus 3 allowable quadruples with a one in the fourth place, $2 + 3 = 5$ of them with a zero in the fourth place, and a total of $h(4) = 8 = F_6$ such quadruples. Similarly the number of quintuples of our desired form with a zero in the fifth place is 5, the number with a one is $3 + 5 = 8$, and the total number of such quintuples is $h(5) = 13 = F_7$. Using mathematical induction, one now easily shows that $h(r) = F_{r+2}$.

REFERENCES

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