ON THE GREATEST PRIMITIVE DIVISORS OF FIBONACCI AND LUCAS NUMBERS WITH PRIME-POWER SUBSCRIPTS

DOV JARDEN, JERUSALEM, ISRAEL

The greatest primitive divisor F'_n of a Fibonacci number F_n is defined as the greatest divisor of F_n relatively prime to every F_x with positive $x < n$.

Similarly, the greatest primitive divisor L'_n of a Lucas number L_n is defined as the greatest divisor of L_n relatively prime to every L_x with nonnegative $x < n$.

The first 20 values of the sequence (F'_n) are:

$F'_1 = 1$, $F'_2 = 1$, $F'_3 = 2$, $F'_4 = 3$, $F'_5 = 5$, $F'_6 = 1$, $F'_7 = 13$,

$F'_8 = 7$, $F'_9 = 17$, $F'_{10} = 11$, $F'_{11} = 89$, $F'_{12} = 1$, $F'_{13} = 233$, $F'_{14} = 29$,

$F'_{15} = 61$, $F'_{16} = 47$, $F'_{17} = 1597$, $F'_{18} = 19$, $F'_{19} = 4181$, $F'_{20} = 41$.

As may be seen from these few examples, the growth of the sequence (F'_n) is very irregular. However, some special subsequences of (F'_n) may occur to be increasing sequences. E.g., the subsequence (F'_p), where p ranges over all the primes, is a strictly increasing sequence (since $F'_p = F_p$ and (F_n) is a strictly increasing sequence beginning with $n = 2$).

Similarly, the subsequence (L'_q), where q ranges over all the odd primes and over all the powers of 2 beginning with 2^2, is a strictly increasing sequence.

The main object of this note is to prove the following inequalities:

(1) $$F'_{p^{x+1}} > F'_{p^x} \quad (p - \text{a prime}, \ x - \text{a positive integer})$$

(2) $$F'_{2p^{x+1}} > F'_{2p^x} \quad (p - \text{a prime}, \ x - \text{a nonnegative integer})$$

(2*) $L'_{p^{x+1}} > L'_{p^x}$ (p – a prime, x – a nonnegative integer)

In other words: the subsequences (F'_{p^x}) and (F'_{2p^x}) of the sequence (F'_n), as well as the subsequence (L'_{p^x}) of the sequence (L'_n), p being a prime and $x = 1, 2, 3, \cdots$, are strictly increasing sequences.

Since (as is well known) the primitive divisors of F_{2n} and L_n $(n \geq 1)$ coincide, we have: $F'_{2n} = L'_n$ $(n \geq 1)$, and especially: $F'_{2^{x+1}} = L'_{2^x}$ $(x \geq 0)$. Hence, (2) and (2*) are equivalent, and, for $p = 2$, also (1) and (2*). Thus it is sufficient to prove (1) for $p \geq 2$ and (2*) for $p \neq 2$.

We shall even show the stronger inequalities:

(3) $F'_{p^{x+1}} > F_{p^x}$ (p – a prime, x – a positive integer)

(3*) $L'_{p^{x+1}} > L_{p^x}$ (p – a prime, x – a nonnegative integer)

Since $F_n \geq F'_n$, $L_n \geq L'_n$, it is obvious that in order to prove (1) for $p \geq 2$, and (2*) for $p \neq 2$, it is sufficient to prove (3) for $p \geq 2$ and (3*) for $p \neq 2$.

The main tools for proving (3) for $p \geq 2$ and (3*) for $p \neq 2$, are the following inequalities:

(4) $F_{n^{x+1}} > F^2_{n^x}$ $(n \geq 2,\ x \geq 1)$

(5) $F_{5^{x+1}} > 5F^2_{5^x}$ $(x \geq 1)$

(4*) $L_{n^{x+1}} > L^2_{n^x}$ $(n \geq 2,\ x \geq 0)$

In order to prove (3) for $p \geq 2$ and (3*) for $p \neq 2$, it is sufficient (as will be shown later) to prove (4) for n a prime ≥ 2 and (4*) for n an odd prime. However, since (4) and (4*) are interesting by themselves, we shall prove them for any positive integer $n \geq 2$.

The following formulae are well known.

(6) $$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n)$$

$$\alpha = \frac{1 + \sqrt{5}}{2} , \quad \beta = \frac{1 - \sqrt{5}}{2}$$

(6*) $$L_n = \alpha^n + \beta^n$$

Since

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \frac{1 + \sqrt{4}}{2} = \frac{3}{2} ,$$

we have:

(7) $$\alpha = \frac{3}{2} .$$

Since

$$\beta = \frac{1 - \sqrt{5}}{2} > \frac{1 - \sqrt{9}}{2} = -1 ,$$

$$\beta = \frac{1 - \sqrt{5}}{2} < \frac{1 - \sqrt{4}}{2} \quad -\frac{1}{2} ,$$

we have

(8) $$-1 < \beta < -\frac{1}{2} , \quad |\beta| < 1 .$$

Since

$$\alpha\beta = \frac{1 + \sqrt{5}}{2} \cdot \frac{1 - \sqrt{5}}{2} = -1 ,$$

we have:

(9) $$\alpha\beta = -1 .$$

For any positive integer $n \geq 3$ we have, by (7):

$$\alpha^{n^{x+1}} = (\alpha^{n^x})^n \geq (\alpha^{n^x})^3 = \alpha^{n^x} \alpha^{2n^x} > \left(\frac{3}{2}\right)^2 \alpha^{2n^x} > 2\alpha^{2n^x} =$$
$$\alpha^{2n^x} + \alpha^{2n^x} > \alpha^{2n^x} + \left(\frac{3}{2}\right)^2 > \alpha^{2n^x} + 3 , \quad \text{whence,}$$

(10) $$\alpha^{n^{x+1}} > \alpha^{2n^x} + 3 \quad (n \geq 3) .$$

For odd $n \geq 3$ we have, by (10), (8), (9):

$$\alpha^{n^{x+1}} - \beta^{n^{x+1}} > \alpha^{2n^x} + 3 = \alpha^{2n^x} + 2 + 1 > \alpha^{2n^x} + 2 + \beta^{2n^x} = (\alpha^{n^x} - \beta^{n^x})^2 \text{, whence}$$

$$(\overline{11}) \qquad \alpha^{n^{x+1}} - \beta^{n^{x+1}} > (\alpha^{n^x} - \beta^{n^x})^2 \quad (2 \nmid n, \; n \geq 3) .$$

For even $n \geq 3$ we have, by (10), (8), (9):

$$\begin{aligned} \alpha^{n^{x+1}} - \beta^{n^{x+1}} &> \alpha^{2n^x} + 3 - \beta^{n^{x+1}} = \alpha^{2n^x} - 2 + (5 - \beta^{n^{x+1}}) \\ &> \alpha^{2n^x} - 2 + \beta^{2n^x} = (\alpha^{n^x} - \beta^{n^x})^2 , \quad \text{whence} \end{aligned}$$

$$(\overline{\overline{11}}) \qquad \alpha^{n^{x+1}} - \beta^{n^{x+1}} > (\alpha^{n^x} - \beta^{n^x})^2 \quad (2 \mid n, \; n \geq 3)$$

Combining $(\overline{11})$ and $(\overline{\overline{11}})$ we have:

$$(11) \qquad \alpha^{n^{x+1}} - \beta^{n^{x+1}} > (\alpha^{n^x} - \beta^{n^x})^2 \quad (n \geq 3) .$$

For $n \geq 3$ we have, by (6), (11):

$$\begin{aligned} F_{n^{x+1}} &= \frac{1}{\sqrt{5}} (\alpha^{n^{x+1}} - \beta^{n^{x+1}}) > \frac{1}{\sqrt{5}} (\alpha^{n^x} - \beta^{n^x})^2 > \left(\frac{1}{\sqrt{5}}\right)^2 (\alpha^{n^x} - \beta^{n^x})^2 \\ &= \left\{ \frac{1}{\sqrt{5}} (\alpha^{n^x} - \beta^{n^x}) \right\}^2 = F^2_{n^x} , \quad \text{whence,} \end{aligned}$$

$$(\overline{4}) \qquad F_{n^{x+1}} > F^2_{n^x} \quad (n \geq 3) .$$

We have, by (6), (9):

$$\begin{aligned} F_{2^{x+1}} &= \frac{1}{\sqrt{5}} (\alpha^{2^{x+1}} - \beta^{2^{x+1}}) > \frac{1}{5} (\alpha^{2^{x+1}} - 2 + \beta^{2^{x+1}}) = \\ &\left\{ \frac{1}{\sqrt{5}} (\alpha^{2^x} - \beta^{2^x}) \right\}^2 = F^2_{2^x} , \quad \text{whence} \end{aligned}$$

$$(\bar{\bar{4}}) \qquad F_{2^{x+1}} > F^2_{2^x} .$$

Combining $(\bar{4})$ and $(\bar{\bar{4}})$ we have (4).

We have, by (7):*

$$\alpha^{5^{x+1}} = (\alpha^{5^x})^5 = (\alpha^{5^x})^3(\alpha^{5^x})^2 > \left(\frac{3}{2}\right)^5 \alpha^{2\cdot 5^x} = \frac{243}{32}\alpha^{2\cdot 5^x} > 7\alpha^{2\cdot 5^x} =$$

$$5\alpha^{2\cdot 5^x} + 2\alpha^{2\cdot 5^x} > 5\alpha^{2\cdot 5^x} + 2\left(\frac{3}{2}\right)^6 > 5\alpha^{2\cdot 5^x} + 22, \quad \text{whence}$$

$$(12) \qquad \alpha^{5^{x+1}} > 5\alpha^{2\cdot 5^x} + 22 .$$

We have, by (12), (8), (9):

$$\alpha^{5^{x+1}} - \beta^{5^{x+1}} > 5\alpha^{2\cdot 5^x} + 22 - \beta^{5^{x+1}} > 5\alpha^{2\cdot 5^x} + 10 + (12 - \beta^{5^{x+1}})$$

$$> 5\alpha^{2\cdot 5^x} + 10 + 5\beta^{2\cdot 5^x} = 5(\alpha^{2\cdot 5^x} + 2 + \beta^{2\cdot 5^x}) = 5(\alpha^{5^x} - \beta^{5^x})^2 ,$$

whence

$$(13) \qquad \alpha^{5^{x+1}} - \beta^{5^{x+1}} > 5(\alpha^{5^x} - \beta^{5^x})^2 .$$

We have by (6), (13), (8):

$$F_{5^{x+1}} = \frac{1}{\sqrt{5}}(\alpha^{5^{x+1}} - \beta^{5^{x+1}}) > \frac{1}{\sqrt{5}} 5(\alpha^{5^x} - \beta^{5^x})^2 > 5\left\{\frac{1}{\sqrt{5}}(\alpha^{5^x} - \beta^{5^x})\right\}^2$$

$$= 5F^2_{5^x} ,$$

whence (5) is valid.

For odd $n \geq 3$ we have, by (6*), (10), (8), (9):

$$L_{n^{x+1}} = \alpha^{n^{x+1}} + \beta^{n^{x+1}} > \alpha^{2n^x} + 3 + \beta^{n^{x+1}} = \alpha^{2n^x} - 2 + (5 + \beta^{n^{x+1}}) =$$

$$\alpha^{2n^x} - 2 + \beta^{2n^x} = (\alpha^{n^x} + \beta^{n^x})^2 = L^2_{n^x} ,$$

whence

*See editorial remark, page 59.

$$(\bar{4}^*) \qquad L_{n^{x+1}} > L^2_{n^x} \quad (2 \nmid n, \quad n \geq 3) .$$

For even $n \geq 3$ we have, by (6*), (10), (8), (9):

$$L_{n^{x+1}} = \alpha^{n^{x+1}} + \beta^{n^{x+1}} > \alpha^{2n^x} + 3 = \alpha^{2n^x} + 2 + 1 > \alpha^{2n^x} + 2 + \beta^{2n^x} = (\alpha^{n^x} + \beta^{n^x})^2 = L^2_{n^x} \,, \quad \text{whence}$$

$$(\bar{\bar{4}}^*) \qquad L_{n^{x+1}} > L^2_{n^x} \quad (2 \mid n, \quad n \geq 3)$$

For $n = 2$ we have the well-known relation: $L_{2^{x+1}} = L^2_{2^x} - 2$, whence

$$(\bar{\bar{\bar{4}}}^*) \qquad L_{2^{x+1}} < L^2_{2^x}$$

Combining $(\bar{4}^*)$, $(\bar{\bar{4}}^*)$ and $(\bar{\bar{\bar{4}}}^*)$ we have (4*).

Proof of (3), (3*).

For $p \neq 5$, $(p, F_{p^x}) = 1$. Hence, by the law of repetition of primes in (F_n), the greatest imprimitive divisor of $F_{p^{x+1}}$ is F_{p^x}, whence, by (4):

$$F'_{p^{x+1}} = F_{p^{x+1}} / F_{p^x} > F_{p^x} \,,$$

i.e., (3) is valid for $p \neq 5$.

For $p = 5$, by the law of repetition of primes in (F_n), the greatest imprimitive divisor of $F_{5^{x+1}}$ is $5F_{5^x}$, whence, by (5):

$$F'_{5^{x+1}} = F_{5^{x+1}} / 5F_{5^x} > F_{5^x} \,,$$

i.e., $F'_{5^{x+1}} > F_{5^x}$, i.e., (3) is valid for $p = 5$.

For $p \neq 2$, by the law of repetition of primes in (L_n), the greatest imprimitive divisor of $L_{p^{x+1}}$ is L_{p^x}, whence, by (4*): $L'_{p^{x+1}} = L_{p^{x+1}} / L_{p^x} > L_{p^x}$, i.e., (3*) is valid for $p \neq 2$.