FIBONACCI SUMMATIONS

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In the first issue of the FIBONACCI QUARTERLY, several problems regarding summation of terms of the Fibonacci series were proposed [1]. They can be solved without too much difficulty by means of intuition followed by mathematical induction. The results for the series suggested in the article "Exploring Fibonacci Numbers" are as follows:

$$\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1 \qquad \sum_{k=1}^{n} F_{4k-3} = F_{2n-1}F_{2n}$$

$$\sum_{k=1}^{n} F_{4k-2} = F_{2n}^2 \qquad \sum_{k=1}^{n} F_{4k-1} = F_{2n}F_{2n+1}$$

$$\sum_{k=1}^{n} F_{4k} = F_{2n+1}^2 - 1 \qquad 2\sum_{k=1}^{n} F_{3k-2} = F_{3n}$$

$$2\sum_{k=1}^{n} F_{3k-1} = F_{3n+1} - 1 \qquad 2\sum_{k=1}^{n} F_{3k} = F_{3n+2} - 1$$

The attempt to extend this work by intuition to such summations as

$$\sum_{k=1}^{n} F_{5k-4}$$

leads to difficulties. One is led therefore to adopt a more mathematical approach in solving the general case of all Fibonacci series summations with subscripts in arithmetic progression, namely,

$$\sum_{k=1}^{n} F_{ak-b}$$

where a and b are positive integers and $b < a$.

We recall that Fibonacci numbers can be given in terms of the roots of the equation $x^2 - x - 1 = 0$ [2]. If these roots are

$$r = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad s = \frac{1 - \sqrt{5}}{2}$$

then

$$F_n = \frac{r^n - s^n}{\sqrt{5}} \quad \text{and} \quad L_n = r^n + s^n ,$$

where F_n is the n^{th} term of the Fibonacci sequence $1, 1, 2, 3, 5, \cdots$ and L_n is the n^{th} term of the Lucas sequence $1, 3, 4, 7, 11, 18, \cdots$. In these terms

$$\sum_{k=1}^{n} F_{ak-b} = \frac{1}{\sqrt{5}} \left[\sum_{k=1}^{n} r^{ak-b} - \sum_{k=1}^{n} s^{ak-b} \right]$$

One can restate the summations on the right-hand side of the equation by using the formula for geometric progressions.

$$\sum_{k=1}^{n} r^{ak-b} = r^{a-b} \left[1 + r^a + r^{2a} + r^{3a} + \cdots + r^{(n-1)a} \right] = r^{a-b} \left(\frac{r^{an} - 1}{r^a - 1} \right)$$

There is an entirely similar formula for the "s" summation. Substituting into the original formula and combining fractions, one obtains

$$\sum_{k=1}^{n} F_{ak-b} = \frac{1}{\sqrt{5}} \left[\frac{s^a r^{an+a-b} - r^a s^{an+a-b} - r^{an+a-b} + s^{an+a-b}}{r^a s^a - r^a - s^a + 1} + \frac{s^a r^{a-b} + r^a s^{a-b} + r^{a-b} - s^{a-b}}{r^a s^a - r^a - s^a + 1} \right]$$

Various simplifications result using the definitions of F_n and L_n in terms of r and s together with the relation $rs = -1$, the product of the roots in the equation $x^2 - x - 1 = 0$ being the constant term -1. For example,

$$s^a r^{an+a-b} - r^a s^{an+a-b} = (rs)^a (r^{an-b} - s^{an-b}) = (-1)^a \sqrt{5}\, F_{an-b} .$$

The denominator can be transformed into $(-1)^a - L_a + 1$. Using these relations the reader may verify without too much difficulty that the final formula is

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b} F_b + F_{a-b}}{(-1)^a + 1 - L_a}$$

With this formula particular cases can be handled with little effort. For example, let $a = 7$, $b = 3$ and $n = 6$. Then

$$\sum_{k=1}^{6} F_{7k-3} = \frac{(-1)^7 F_{39} - F_{46} + (-1)^4 F_3 + F_4}{(-1)^7 + 1 - L_7}$$

$$= \frac{-63245986 - 1836311903 + 2 + 3}{-29}$$

$$= 65501996 .$$

This result may be checked by actually summing the series:

$$F_4 + F_{11} + F_{18} + F_{25} + F_{32} + F_{39} \text{ or } 3 + 89 + 2584 + 75025 + 2178309 + 63245986$$

the result being 65501996.

REFERENCES

1. Brother U. Alfred, Exploring Fibonacci Numbers, Fibonacci Quarterly, Vol. 1, No. 1, February 1963, pp. 57–64.
2. I. Dale Ruggles, Some Fibonacci Results Using Fibonacci-Type Sequences, Fibonacci Quarterly, Vol. 1, No. 2, April 1963, pp. 75–80.