ON THE ORDERING OF FIBONACCI SEQUENCES

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We may define a Fibonacci sequence by taking any two relatively prime integers and employing the relation

$$f_n = f_{n-1} + f_{n-2}$$

to extend the sequence to subscripts going to plus infinity or subscripts going to minus infinity. Evidently, since any two successive terms of a given sequence define the sequence, there appears to be at first glance an element of confusion in the situation. How may this be obviated and how, once it is removed, is it possible to arrange Fibonacci sequences according to some rational order? Such are the questions that will be answered in this short paper.

It is a remarkable fact that every Fibonacci sequence has two parts: the one going to the right with all the signs the same may be called the monotonic portion; the other going to the left with signs alternating may be denoted the alternating portion. The sequence will be designated positive or negative according as the monotonic portion has all terms plus or minus respectively. However, since a negative sequence may be obtained from a positive sequence by changing the signs of all terms, it will be sufficient in what follows to consider the positive sequence.

Starting, then, with two positive terms a and b with $a < b$, we work back to $a' = b - a$; if this is less than a, we next form $a'' = a - a'$; and so on. Evidently, this process cannot be continued indefinitely and so we finally arrive at a term which is greater than the term which follows it. Once this occurs, the next term to the left is negative and from then on the signs alternate.

It is important to note that in this process we have arrived at a smallest positive term which has the characteristic that it is less than one-half the following positive term. This property of the smallest positive term is unique in the monotonic portion of the sequence. Let us call this smallest non-negative, f_0, and the subsequent term, f_1. We thus have an unambiguous means of representing the Fibonacci sequence by giving these two terms: (f_0, f_1).

If we had started with a positive term and a negative term, as long as terms alternate in sign in going to the right in the sequence, the absolute values decrease in magnitude. Since this cannot go on indefinitely, there must come a term which is of the same sign as the preceding term. From then on the sequence is monotonic. Evidently, the second term in the monotonic portion of the sequence is a minimum for that part of the sequence and so once more it is possible to represent the sequence according to the minimum term and the term that follows it.

Now that a unique representation of each Fibonacci sequence has been achieved, it might appear desirable to have some method of arranging these sequences in order. One means of doing so is by the use of the quantity

$$D = f_1 f_{-1} - f_0^2 = f_1^2 - f_1 f_0 - f_0^2$$

which is characteristic of any given sequence. Intuitively it appears that for any sequence

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n D$$

Suppose this to be true to n. Then

$$f_{n+2} f_n - f_{n+1}^2 = f_{n+1} f_n + f_n^2 - f_{n+1}^2 = f_n^2 - f_{n+1} f_{n-1} = (-1)^{n+1} D$$

so that the formula is seen to hold by mathematical induction.

For any given value of f_1, since

$$D = f_1^2 - f_0 (f_1 + f_0)$$

and since

$$f_0 < f_1 / 2 \quad ,$$

it follows that

$$D > f_1^2 - f_1 (f_1 + f_1/2) / 2$$

or

$$D > f_1^2 / 4$$

Accordingly, by considering successive values of f_1 and the various Fibonacci sequences that may be associated with these values, it is possible to arrive at certain knowledge regarding the Fibonacci sequences that may be associated with allowed values of D. This information is summarized for values of D up to 1000 in the following table.

TABLE OF FIBONACCI SEQUENCES
HAVING A GIVEN VALUE OF D

D	SEQUENCES	D	SEQUENCES
1	(0,1)	431	(5,24), (14,33)
5	(1,3)	439	(6,25), (13,32)
11	(1,4), (2,5)	445	(7,26), (12,31)
19	(1,5), (3,7)	449	(8,27), (11,30)
29	(1,6), (4,9)	451	(3,23), (9,28), (10,29), (17,37)
31	(2,7), (2,8)	461	(1,22), (20,41)
41	(1,7), (5,11)	479	(2,23), (19,40)
55	(1,8), (6,13)	491	(7,27), (13,33)
59	(2,9), (5,12)	499	(9,29), (11,31)
61	(3,10), (4,11)	505	(1,23), (21,43)
71	(1,9), (7,15)	509	(4,25), (17,38)
79	(3,11), (5,13)	521	(5,26), (16,37)
89	(1,10), (8,17)	541	(3,25), (19,41)
95	(2,11), (7,16)	545	(8,29), (13,34)
101	(4,13), (5,14)	551	(1,24), (10,31), (11,32), (22,45)
109	(1,11), (9,19)	569	(5,27), (17,39)
121	(3,13), (7,17)	571	(2,25), (21,44)
131	(1,12), (10,21)	589	(3,26), (7,29), (15,37), (20,43)
139	(2,13), (9,20)	599	(1,25), (23,47)
145	(3,14), (8,19)	601	(9,31), (13,35)
149	(4,15), (7,18)	605	(4,27), (19,42)
151	(5,16), (6,17)	619	(5,28), (18,41)
155	(1,13), (11, 23)	631	(6,29), (17,40)
179	(5,17), (7,19)	641	(7,30), (16,39)
181	(1,14), (12,25)	649	(1,26), (8,31), (15,38), (24,49)
191	(2,15), (11,24)	655	(9,32), (14,37)
199	(3,16), (10,23)	659	(10,33), (13,36)
205	(4,17), (9,22)	661	(11,34), (12,35)
209	(1,15), (5,18), (8,21), (13,27)	671	(2,27)(5,29), (19,43), (23,48)
211	(6,19), (7,20)	691	(3,28), (22,47)
229	(3,17), (11,25)	695	(7,31), (17,41)
239	(1,16), (14,29)	701	(1,27), (25,51)
241	(5,19), (9,23)	709	(4,29), (21,46)
251	(2,17), (13,28)	719	(11,35), (13,37)
269	(4,19), (11,26)	739	(6,31), (19,44)
271	(1,17), (15,31)	745	(3,29), (23,49)
281	(7,22), (8,23)	751	(7,32), (18,43)
295	(3,19), (13,29)	755	(1,28), (26,53)
401	(7,25), (11,29)	761	(8,33), (17,42)
409	(3,22), (16,35)	769	(9,34), (16,41)
419	(1,21), (19,39)	779	(2,29), (11,36), (14,39), (25,52)
421	(4,23), (15,34)	781	(5,31), (12,37), (13,38), (21,47)

D	SEQUENCES	D	SEQUENCES
809	(7,33), (19,45)	905	(11,38), (16,43)
811	(1,29), (27,55)	911	(13,40), (14,41)
821	(4,31), (23,50)	919	(3,32), (26,55)
829	(9,35), (17,43)	929	(1,31), (29,59)
839	(5,32), (22,49)	941	(4,33), (25,54)
841	(11,37), (15,41)	955	(9,37), (19,47)
859	(3,31), (25,53)	961	(5,34), (24,53)
869	(1,30), (7,34), (20,47), (28,57)	971	(11,39), (17,45)
881	(8,35), (19,46)	979	(6,35), (13,41), (15,43), (23,52)
895	(2,31), (27,56)	991	(1,32), (30,61)
899	(5,33), (10,37), (17,44), (23,51)	995	(7,36), (22,51)

By adopting the convention that for several sequences having the same value of D, the ordering will be determined by which has the smaller value of f_0, it becomes possible to give the Fibonacci sequences a precise arrangement. The first few members would be as follows:

$$S_1(0,1),\quad S_2(1,3),\quad S_3(1,4),\quad S_4(2,5),\quad S_5(1,5),\quad S_6(3,7),\quad S_7(1,6),\quad S_8(4,9),\quad \text{etc.}$$

The above approach in representing Fibonacci sequences and ordering them is all by way of suggestion. There are doubtless other ways of achieving the same objective. It would be very helpful if additional proposals were aired before a final standard is adopted.

FURTHER APPEARANCE OF THE FIBONACCI SEQUENCE

(Cont. from p. 42)

for the classicist, no less than for the historian of Mathematics. "Measure and symmetry," observed Socrates, "are beauty and virtue all the world over."

REFERENCES

1. S. L. Basin, "The Fibonacci Sequence as it Appears in Nature," Fibonacci Quarterly 1(1), 1963, 53 – 56.
2. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math Monthly, 68(5), 1961, 455–459.
3. J. Hambidge, "Dynamics Symmetry," Yale University Press, 1920.
4. G. E. Duckworth, "Structural Patterns and Proportions in Vergil's Aeneid," Univ. of Michigan Press, 1962.