DYING RABBIT PROBLEM REVIVED

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In the first issue of the Fibonacci Quarterly the following problem was proposed [1]. Suppose that in the original Fibonacci rabbit breeding problem, we allow for the dying of rabbits. Those that are bred in February, for example, begin to breed in April and continue breeding monthly through February of the following year. At the end of this month they die. What would be the formula for the number of pairs of rabbits at the end of n months for $n \geq 13$?

Originally, it was thought that the rabbits removed would constitute a sequence which could be readily identified with an expression involving Fibonacci numbers. But after several attempts by a number of people it appeared that it would be difficult to arrive at an answer by straightforward intuition. The following development will indicate why this is so.

First of all we shall set down a table showing how the rabbits propagate over a two-year period. It will be noted that the original table values for the case in which rabbits do not die are positive while a negative term is introduced to show the effect of allowing rabbits to die.

n	Breeding Rabbits	Non-Breeding Rabbits	Bred Rabbits	Dying Rabbits	Rabbits End of Month
1	0	1	0	0	1
2	1	0	1	0	2
3	1	1	1	0	3
4	2	1	2	0	5
5	3	2	3	0	8
6	5	3	5	0	13
7	8	5	8	0	21
8	13	8	13	0	34
9	21	13	21	0	55
10	34	21	34	0	89
11	55	34	55	0	144
12	89	55	89	-1	233 - 1
13	144 - 1	89	144 - 1	0	377 - 2
14	233 - 1	144 - 1	233 - 1	-1	610 - 4
15	377 - 3	233 - 1	377 - 3	-1	987 - 8
16	610 - 5	377 - 3	610 - 5	-2	1597 - 15
17	987 - 10	610 - 5	987 - 10	-3	2584 - 28
18	1597 - 18	987 - 10	1597 - 18	-5	4181 - 51
19	2584 - 33	1597 - 18	2584 - 33	-8	6765 - 92
20	4181 - 59	2584 - 33	4181 - 59	-13	10946 - 164

n	Breeding Rabbits	Non-Breeding Rabbits	Bred Rabbits	Dying Rabbits	Rabbits End of Month
21	6765 - 105	4181 - 59	6765 - 105	-21	17711 - 290
22	10946 - 185	6765 - 105	10946 - 185	-34	28657 - 509
23	17711 - 324	10946 - 185	17711 - 324	-55	46368 - 888
24	28657 - 564	17711 - 324	28657 - 564	-89	75025 - 1541

For the sake of convenience, a table of the negative values is formed with a shift of numbering, the first row in the new table corresponding to n = 14 in the old.

n	a_n	a_{n-1}	a_n	F_n	T_n
1	1	1	1	1	4
2	3	1	3	1	8
3	5	3	5	2	15
4	10	5	10	3	28
5	18	10	18	5	51
6	33	18	33	8	92
7	59	33	59	13	164
8	105	59	105	21	290
9	185	105	185	34	509
10	324	185	324	55	888
11	564	324	564	89	1541

The following relations may be noted apart from those implicit in the column headings.

$$a_{n+1} = a_n + a_{n-1} + F_n$$

$$T_n = 2a_n + a_{n-1} + F_n$$

Using the relation for a_{n+1} we obtain the following succession of relations.

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= a_1 + a_0 + F_1 = 2 + F_1 \\
a_3 &= a_2 + a_1 + F_2 = 3 + F_1 + F_2 \\
a_4 &= a_3 + a_2 + F_3 = 5 + 2F_1 + F_2 + F_3 \\
a_5 &= a_4 + a_3 + F_4 = 8 + 3F_1 + 2F_2 + F_3 + F_4 \\
a_6 &= a_5 + a_4 + F_5 = 13 + 5F_1 + 3F_2 + 2F_3 + F_4 + F_5 \\
a_7 &= a_6 + a_5 + F_6 = 21 + 8F_1 + 5F_2 + 3F_3 + 2F_4 + F_5 + F_6
\end{aligned}$$

It is clear that a formula involving Fibonacci numbers is emerging. For example, a_7 can be written:

$$a_7 = F_8 + F_6F_1 + F_5F_2 + F_4F_3 + F_3F_4 + F_2F_5 + F_1F_6$$

and in general, it could be shown by mathematical induction that:

$$a_n = F_{n+1} + \sum_{k=1}^{n-1} F_kF_{n-k} \; .$$

The problem then reduces to finding a formula for the summation on the right. Using the roots r and s of the equation $x^2 - x - 1 = 0$ in terms of which:

$$F_n = \frac{r^n - s^n}{\sqrt{5}} \qquad \text{and} \qquad L_n = r^n + s^n$$

where

$$r = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad s = \frac{1 - \sqrt{5}}{2}$$

we have

$$F_kF_{n-k} = \frac{(r^k - s^k)(r^{n-k} - s^{n-k})}{5}$$

$$= \frac{r^n + s^n - r^ks^k(r^{n-2k} + s^{n-2k})}{5}$$

But $r^ks^k = (-1)^k$ since rs, the product of the roots of the given equation, is the constant term -1. Thus

$$F_kF_{n-k} = \frac{L_n + (-1)^{k+1}L_{n-2k}}{5}$$

This is the expression that must be summed from 1 to $n - 1$ over k. However, since the first part $L_n/5$ does not involve k, it is essentially a constant taken $n - 1$ times so that this part of the sum becomes $(n - 1)L_n/5$. The second half is

$$\frac{1}{5}\sum_{k=1}^{n-1} (-1)^{k+1} L_{n-2k} = \frac{1}{5}\Big[L_{n-2} - L_{n-4} + L_{n-6} - L_{n-8} + \cdots + (-1)^n L_{-n+2}\Big]$$

$$= \frac{1}{5}\Big[F_{n-1} + F_{n-3} - F_{n-3} - F_{n-5} + F_{n-5} + F_{n-7} - \cdots + (-1)^nF_{-n+3} + (-1)^nF_{-n+1}\Big]$$

It will be noted that all terms cancel out except the first and last and since $(-1)^n F_{-n+1} = F_{n-1}$, the total of the summation is $2F_{n-1}/5$. Thus the value of a_n is given by the expression

$$a_n = F_{n+1} + \frac{(n-1)}{5} L_n + \frac{2F_{n-1}}{5} .$$

Substituting $L_n = F_{n+1} + F_{n-1}$, this can be transformed to

$$a_n = \frac{1}{5}\left[(n+4)F_{n+1} + (n+1)F_{n-1}\right]$$

As a check, for $n = 7$, this becomes

$$1/5\,[11 \cdot 21 + 8 \cdot 8] = 59$$

After suitable transformations one can find a value of T_n equal to

$$1/5\left[(3n+10)F_{n+1} + (n+6)F_n\right]$$

Reconverting back to our original notation, the solution of the dying rabbit problem can be expressed as follows $(n \geq 13)$:

$$F_{n+1} - 1/5\left[(3n-29)F_{n-12} + (n-7)F_{n-13}\right]$$

REFERENCE

1. Brother U. Alfred, Exploring Fibonacci Numbers, The Fibonacci Quarterly, Vol. 1, No. 1, Feb. 1963, pp. 57—63.