FIBONACCI NUMBERS, CHEBYSHEV POLYNOMIALS GENERALIZATIONS AND DIFFERENCE EQUATIONS

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INTRODUCTION

In order to consider Fibonacci numbers, generalized Fibonacci numbers, Chebyshev polynomials, and other related sequences all under one heading we will discuss the sequences generated by the homogeneous linear second order difference equation with constant coefficients,

$$u_0;\ u_1;\ u_{n+1} = a u_n + b u_{n-1}, \quad \text{for } n \geq 1. \tag{1}$$

First we note how the special cases arise. If $a = b = 1$, then the generalized Fibonacci numbers, H_n, discussed by Horadam [2] are produced. Further specialization leads to Fibonacci numbers, F_n, for $u_0 = 0$, $u_1 = 1$; to Lucas numbers, L_n, for $u_0 = 2$, $u_1 = 1$. If a and b are polynomials in x, then a sequence of polynomials is generated. In particular, if $a = 2x$ and $b = -1$, then we have Chebyshev polynomials [1:10.11] — of the first kind, $T_n(x)$, for $u_0 = 1$, $u_1 = x$; of the second kind, $U_n(x)$, for $u_0 = 1$, $u_1 = 2x$.

FIBONACCI-CHEBYSHEV RELATIONS

Since the same difference equation can be used to generate these entities, by an appropriate interpretation of a, b, u_0, and u_1, one then expects relationships to exist between some of them. The Fibonacci and Lucas numbers are related to the Chebyshev polynomials by the equations

$$2 i^{-n} T_n(i/2) = L_n; \quad i^{-n} U_n(i/2) = F_{n+1}.$$

The second of these can be obtained, for example, by considering

$$U_0(x) = 1; \quad U_1(x) = 2x; \quad U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x),$$

substituting $i/2$ for x, and multiplying by i^{-n-1} so that we have

$$U_0(i/2) = 1; \quad i^{-1}U_1(i/2) = 1; \quad i^{-n-1}U_{n+1}(i/2) = i^{-n}U_n(i/2) + i^{-n+1}U_{n-1}(i/2),$$

which is the same as the Fibonacci sequence,

$$F_1 = 1; \quad F_2 = 1; \quad F_{n+1} = F_n + F_{n-1}, \quad \text{for } n \geq 2 .$$

This close relation leads one to investigate sources for Chebyshev polynomials in order to try to find not too familiar relations involving Fibonacci and Lucas numbers, and vice versa. One such standard source for identities involving Chebyshev polynomials is Erdelyi, et al. [1:10.9, 10.11]. Most of the results which can be obtained were known as early as Lucas [3]; in fact, much of his discussion contains relations with trigonometric identities which lead, of course, to Chebyshev polynomial identities, since

$$T_n(\cos\theta) = \cos n\theta, \quad U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta .$$

Some examples of such pairs of relations follow.

$$U_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (n-m)!}{m!\,(n-2m)!}(2x)^{n-2m}, \qquad [1:10.11\ (23)].$$

$$F_n = \sum_{m=0}^{[n/2]} \binom{n-m}{m}, \qquad [3:(72)].$$

$$T_n(x) = \frac{n}{2}\sum_{m=0}^{[n/2]} \frac{(-1)^m (n-m-1)!}{m!\,(n-2m)!}(2x)^{n-2m} \qquad [1:10.11\ (22)].$$

$$L_n = \sum_{m=0}^{[n/2]} \frac{n}{n-m}\binom{n-m}{m}.$$

Examples of interesting generating functions are given by [1:10.11 (32), (33)] which for $x = i/2$, $z = -iu$ lead to

$$(2) \quad 2^{-\frac{1}{2}}(1-u-u^2)^{-\frac{1}{2}}\{1 - u/2 + (1-u-u^2)^{\frac{1}{2}}\}^{-\frac{1}{2}} = u^{-1}\sum_{n=0}^{\infty} 2^{-2n}\binom{2n}{n}F_n u^n$$

(3) $$2^{-\frac{1}{2}}(1-u-u^2)^{-\frac{1}{2}}\{1-u/2+(1-u-u^2)^{\frac{1}{2}}\}^{\frac{1}{2}} = \sum_{n=0}^{\infty} 2^{-2n-1}\binom{2n}{n} L_n u^n \quad .$$

If the series (2) and (3) are multiplied together as power series, then we have

$$2^{-1}(1-u-u^2)^{-1} = u^{-1}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 2^{-2n-1}\frac{(2k)!\,(2n-2k)!}{k!\,k!\,(n-k)!\,(n-k)!} L_k F_{n-k}\right) u^n \quad .$$

However, this is a generating function for F_n,

$$2^{-1}(1-u-u^2)^{-1} = u^{-1}\sum_{n=0}^{\infty}(F_n/2)\,u^n \quad ,$$

so that by equating coefficients and rearranging somewhat we obtain

$$\sum_{k=0}^{n}\binom{n}{k}L_k\binom{n}{n-k}F_{n-k}\Big/\binom{2n}{2k} = 2^{2n}F_n\Big/\binom{2n}{n} \quad .$$

Two examples of explicit formulas can be obtained by substituting $\lambda = 1$, $x = i/2$ into the second forms of [1: 10.9 (21), (22)], since $C_n^1(x) = U_n(x)$, and simplifying.

$$F_{2m+1} = (-1)^m \sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{m-k}(-5)^k \quad ;$$

$$F_{2m+2} = (-1)^m \sum_{k=0}^{m}\binom{m+k+1}{m-k}(-5)^k \quad .$$

IDENTITIES FOR THE DIFFERENCE EQUATION

In general, the solution to the linear difference equation can be written

(4) $$u_n = \{z_2^n(u_1 - z_1u_0) - z_1^n(u_1 - z_2u_0)\}/(z_2 - z_1)$$

provided $z_2 \neq z_1$ are the roots of the characteristic equation $z^2 - az - b = 0$.

(A suitable modification can be made for $z_2 = z_1$ by a passage to the limit — the formulas must be altered appropriately.) An interesting method of arriving at this is given by I. Niven and H. S. Zuckerman [4: pp 90 — 92]. (This method can be extended to higher order difference equations and to non-homogeneous equations. Further, it has an analog for differential equations.) If the z's are expressed in terms of a and b and the resulting binomials are expanded, then an alternate form of considerable use is obtained,

$$(5)\qquad u_n = 2^{-n} u_0 \sum_{k=0}^{[n/2]} \binom{n}{2k} a^{n-2k} (a^2 + 4b)^k + 2^{-n}(2u_1 - au_0) \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} a^{n-1-2k} (a^2+4b)^k$$

Here we can define sequences from the sums in (5), for $n \geq 0$. Let $\varphi_0 = 0$, $\varphi_1 = 1$; $\lambda_0 = 2$, $\lambda_1 = a$ so that

$$(6)\qquad \varphi_n = 2^{-n+1} \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} a^{n-1-2k} (a^2 + 4b)^k ,$$

$$(7)\qquad \lambda_n = 2^{-n+1} \sum_{k=0}^{[n/2]} \binom{n}{2k} a^{n-2k} (a^2 + 4b)^k ,$$

which correspond, respectively, to the Fibonacci and Lucas numbers. The general sequence, u_n, can then be written as a linear combination of these; i.e.,

$$(8)\qquad u_n = \frac{1}{2} u_0 \lambda_n + \frac{1}{2}(2u_1 - au_0)\varphi_n .$$

Since also from (4) we can write

$$\varphi_n = (z_2^n - z_1^n)/(z_2 - z_1) ,$$

$$\lambda_n = \{z_2^n(a - 2z_1) - z_1^n(a - 2z_2)\}/(z_2 - z_1) ,$$

and since $z_1 z_2 = -b$, a relation between the φ's and λ's can be obtained,

(9) $$\lambda_n = a\varphi_n + 2b\varphi_{n-1} .$$

This generalizes a known formula, $L_n = F_n + 2F_{n-1}$, relating the Lucas and Fibonacci numbers. The companion expression, $5F_n = L_n + 2L_{n-1}$, becomes

$$(a^2 + 4b)\varphi_n = a\lambda_n + 2b\lambda_{n-1} .$$

These can then be used to express u_n in terms of either the φ's or the λ's;

(10) $$u_n = u_1\varphi_n + bu_0\varphi_{n-1} ,$$

$$(a^2 + 4b)u_n = (2bu_0 + au_1)\lambda_n + b(2u_1 - au_0)\lambda_{n-1} .$$

One point of interest is that the list of identities given by Horadam [2] for his generalized Fibonacci numbers, H_n, (u_0, u_1 arbitrary; $a = b = 1$) will yield an analogous list for the general case, with suitable modifications of his formula (1), and with the exception of his formula (16). This latter, "Pythagorean relation," is based upon the identity

$$H_{n+3}^2 - 4H_{n+1}H_{n+2} - H_n^2 = 0$$

for which the analog is

$$u_{n+3}^2 - a(3b + a^2)u_{n+1}u_{n+2} - b^3u_n^2 = (-b)^{n+1}e(b^2 - a) ,$$

where

(11) $$e = u_1^2 - au_1u_0 - bu_0^2 .$$

Unless this extra term is zero; i.e., unless $b^2 = a$ or $u_0z = u_1$, the Pythagorean relation does not generalize. In the set of identities for the general equation the special case φ_n introduced in (6) plays the same role with respect to the u_n as do the Fibonacci numbers with respect to the H_n. For example, (10) provides an extension of Horadam's (7); i.e., if $a = b = 1$ so that $u_n = H_n$ and replacing n by $r + 1$, then (10) becomes

$$H_{r+1} = H_0F_r + H_1F_{r+1} \quad .$$

Two further examples of how one can generalize Horadam's formulas follow. We consider his (8) and (12), several of the others being special cases of these.

$$(8) \qquad H_{n+r} = H_{n-1}F_r + H_nF_{r+1} \ ;$$

$$(12) \qquad H_nH_{n+r+1} - H_{n-s}H_{n+r+s+1} = (-1)^{n-s}eF_sF_{r+s+1} \quad .$$

The general expressions are

$$(12) \qquad u_{n+r} = bu_{n-1}\varphi_r + u_n\varphi_{r+1} \quad ,$$

$$(13) \qquad u_nu_{n+r+1} - u_{n-s}u_{n+r+s+1} = (-b)^{n-s}e\varphi_s\varphi_{r+s+1} \quad ,$$

where e is defined by (11); φ_n, by (6).

Proof of (12). We can write, using (10)

$$u_{n+1} = a(u_1\varphi_n + bu_0\varphi_{n-1}) + b(u_1\varphi_{n-1} + bu_0\varphi_{n-2})$$

and then replace $b\varphi_{n-2}$ by $\varphi_n - a\varphi_{n-1}$ and $au_1 + bu_0$ by u_2 to obtain

$$u_{n+1} = u_2\varphi_n + bu_1\varphi_{n-1} \quad .$$

Hence, by induction, the generalization is obtained. The substitution of $r + 1$ for n and $n - 1$ for r with $a = b = 1$ reduces this to the case for H_n's.

Proof of (13). If the appropriate expressions from (4) are substituted into the left side of this equation, and the result is simplified, the right side can then be obtained. Other formulas can sometimes be generalized in the same manner.

The analog to Horadam's (13),

$$b^2u_n^3 + au_{n+1}^3 - (a^2 + b)u_nu_{n+1}^2 = (-b)^ne(au_{n+1} - bu_n) \quad ,$$

is more complicated. It reduces to

$$H_n^3 + H_{n+1}^3 = 2H_nH_{n+1}^2 + (-1)^n eH_{n-1} \quad .$$

We note here the misprint; H_{n-1} was omitted.

GENERALIZED CHEBYSHEV POLYNOMIALS

In (1) let a, b, u_0, u_1 represent polynomials in x. Then u_n becomes a polynomial in x and the various formulas (5) — (13) can be interpreted as formulas involving polynomials. From (8) we note that these polynomials $u_n(x)$ can be expressed in terms of "Fibonacci," $\varphi_n(x)$, and "Lucas," $\lambda_n(x)$, polynomials. The polynomials $\varphi_n(x)$ now play the same special role as the numbers φ_n; for example, formula (12) becomes

$$u_{n+1}(x) = b(x)u_{n-1}(x)\varphi_r(x) + u_n(x)\varphi_{r+1}(x) \quad .$$

The special case $a(x) = 2x$, $b(x) = -1$ leads to the set of polynomials, $H_n(x)$, corresponding to the numbers H_n. We then have analogously from (8),

$$H_n(x) = H_0(x)T_n(x) + (H_1(x) - xH_0(x))U_{n-1}(x) \quad ,$$

where $T_n(x) = \frac{1}{2}\lambda_n(x)$ and $U_{n-1}(x) = \varphi_n(x)$ are again the Chebyshev polynomials. Other identities can be written by inspection from Horadam's list for these "generalized Chebyshev" polynomials.

We note finally that a generating function can be obtained in the usual manner. One assumes a form $g(x, z) = \Sigma u_n(x)z^n$ and obtains a relation by using the difference equation. For the polynomials $u_n(x)$ this is

$$g_n(x,z) = \{u_0(x) + (u_1(x) - a(x)u_0(x))z\}\{1 - a(x)z - b(x)z^2\}^{-1} \quad .$$

Hence the special cases $\lambda_n(x)$ and $\varphi_n(x)$ can be generated from

$$g_\lambda(x,z) = \{2 - a(x)z\}\{1 - a(x)z - b(x)z^2\}^{-1} \quad ,$$

$$g_\varphi(x,z) = z\{1 - a(x)z - b(x)z^2\}^{-1} \quad .$$

REFERENCES

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