# FIBONACCI NIM

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The term Nim refers to any mathematical game in which two players remove objects from one or more piles. Fibonacci Nim was invented by Dr. R. E. Gaskell of Oregon State University, and is a variation of One Pile [1]. In One Pile, two players alternately remove at least $a$, but no more than $q$ objects from a pile of $n$ objects, the winner being the player who removes the last object, where $n$ is a variable integer, and $a$ and $q$ are predetermined integral constants. The strategy is to leave your opponent a situation where $n \equiv 0$ modulo $(a + q)$. This is a "safe position." When $n \equiv i$ modulo $(a + q)$ where $i \not\equiv 0$, the position is "unsafe."

An unsafe [2] position is defined as one in which at least one winning move is possible. A safe position is one in which there are no winning moves possible and every move on this position must make the position unsafe.

In Fibonacci Nim, the determination of safe and unsafe positions is slightly more complex than in One Pile.

## RULES OF THE GAME

The rules of Fibonacci Nim are the same as in One Pile with $a = 1$; but $q$, a constant in One Pile, is a variable in this game. On the first move, $q_1$ is equal to $n - 1$. After the first move, $q_m$ is equal to twice the number of objects removed by the opponent on the $(m - 1)$th move. Let $r_{m-1}$ be the number of objects removed by a player on the $(m - 1)$th move. Then: $q_m = 2r_{m-1}$. For example, if $n = 16$, on the first move player A may remove up to 15. If he removes 3, player B may remove as many as 6, since $q_2 = 2r_1 = 2 \cdot 3 = 6$. If player B removes 4, then A may remove as many as 8, and so on.

## STRATEGY

As in all Nim games, the strategy of Fibonacci Nim calls for the determination of safe positions. The simplest way to determine whether any given situation is safe is to first represent the number of objects left in a Fibonacci number system.*

*See comment No. 1 at the end of this article.

To represent a given number n in the binary system, the binary sequence is used, where $b_n = b_{n-i} + b_{n-1}$, and $b_1$ is defined as 1. Let the Fibonacci sequence be defined in the following way: $f_n = f_{n-1} + f_{n-2}$, where $f_{-1}$ is defined as 0 and $f_0$ as 1. $f_1$ through $f_6$ are then determined as 1, 2, 3, 5, 8 and 13.

It is generally known that by using either a 1 or a 0 in the nth digit from the left of the decimal point to represent the presence or absence of $b_n$, any number may be represented. Similarly, using $f_1$, $f_2$, etc. in place of $b_1$, $b_2$, etc., any number may be represented in a Fibonacci number system, if one remembers to start marking the largest digits first. Thus, $8_{ten}$ is always represented as $10000_f$ and never as $1100_f$ or $1011_f$. Notice that using this rule not only makes the representation of any given number unique, it also makes it impossible for two 1's to appear in a number without at least one 0 separating them.*

In the representation of any number $n > 0$ in the Fibonacci number system, there must be at least one 1. Let the 1 that is farthest to the right on the mth move be $F_m$. If $n = 19_{ten} = 101001_f$, $F = f_1 = 1_{ten}$. If $n = 18_{ten} = 101000_f$, $F = f_4 = 5$. If, on the mth move, $q_m < F_m$, the situation is safe. If $q_m \geq F_m$, the situation is unsafe, and the winning move is to remove exactly $F_m$ objects. For example, if on the first move $n = 10_{ten} = 10010_f$, $q_1 = 9$ and $F_1 = 2$. Since $q_1 > F_1$, the situation is unsafe and the winning move is to remove exactly 2 objects. If player A removes 2 objects, then for player B, $n = 8_{ten} = 10000_f$, $q_2 = 2r_1 = 4$, and $F_2 = 8$. Since $q_2 < F_2$, the situation is safe, and player B will lose unless player A makes a mistake.

## PROOF

To prove the strategy correct, it must be proven that unsafe positions can always be made safe and that safe positions can only be made unsafe.

## FIRST RESULT

Any unsafe position can be made safe.

By definition, on the mth move, $F_m$ can be removed from an unsafe position. If $F_m = n$, then by removing $F_m$ objects the game is automatically won. If $n > F_m$ then, from the definition of $F_m$, there is another 1, which is the second 1 from the right. Let the Fibonacci number that this 1 repre-

*See comment No. 2 at the end of this article.

sents be $f_k$. Let the Fibonacci number that $F_m$ represents be $f_i$. It has already been shown that between any two 1's in a Fibonacci representation of a number, that there must be at least one 0. It follows that there is at least one Fibonacci number greater than $f_i$, but less than $f_k$. Let $f_j$ be the next Fibonacci number after $f_i$. It may or may not be the immediate predecessor of $f_k$.

$$f_i < f_j$$

$$2f_i < f_j + f_i$$

$$2f_i < f_k$$

$$q_{m+1} = 2F_m = 2f_i$$

$$q_{m+1} > f_k$$

But $f_k = F_{m+1}$, after $F_m$ has been removed.

$$q_{m+1} < F_{m+1}$$

Thus, by removing $F_m$ objects from an unsafe position on the mth move, the position will be safe on the (m + 1)th move.

SECOND RESULT

Any move from a safe position must make it unsafe.

Since any move on a safe position on the mth move can never take as many as $F_m$ objects, it follows that $F_{m+1} < F_m$. Let n on the mth move equal $c + F_m = c + f_i$. Let n on the (m + 1)th move equal $c + c_1 + F_{m+1} = c + c_1 + f_h$. Suppose $c_1 + f_h$ can be written in the form $f_{i-1} + f_{i-3} + f_{i-5} \cdots f_{h+2} + f_h$. If $f_i$ is written $1000000 \cdots$, i.e., a 1 followed by $i - 1$ 0's, then $c_1 + f_h$ is written $101010 \cdots 101$ followed by enough 0's to make $i - 1$ digits. The last 1, by definition, represents $f_h$. Let $f_f + f_g$ be the two immediate predecessors of $f_h$. If $f_g$ is added to $c_1 + f_h$, it is found that:

$$\begin{array}{r} 101010 \cdots\ 101000 \cdots \\ +\ 100 \cdots \\ \hline 1000000 \cdots\ 000000 \cdots \end{array}$$

In other words, $c_1 + f_h + f_g = f_i$.* If $c_1 + f_h$ is less than $f_{i-1} + f_{i-3} + f_{i-5} \cdots + f_{h+2} + f_h$, it follows that $c_1 + f_h + f_g < f_i$. Therefore:

*See comment No. 3 at the end of this article.

$$r_m = f_i - (c_1 + f_h) \geq f_g \tag{1}$$

This means that any move that leaves $f_h$ as $F_{m+1}$ must remove at least $f_g$ objects.

$$f_g \geq f_f$$

$$2f_g \geq f_f + f_g$$

$$2f_g \geq f_h \tag{2}$$

$$q_{m+1} = 2r_m$$

$$q_{m+1} \geq 2f_g \quad \text{(by equation (1))}$$

$$q_{m+1} \geq f_h \quad \text{(by equation (2))}$$

But

$$f_h = F_{m+1} \; .$$

$q_{m+1} \geq F_{m+1}$, and the position is unsafe. Thus any move on a safe position makes it unsafe.

## GENERALIZED FIBONACCI NIM

Suppose $q_m = r_{m-1}$. Then the binary system will determine $F_m$ (or more correctly, $B_m$). Safe and unsafe positions will be determined in exactly the same way, and the proof parallels the one given above. If the binary sequence is called a Fibonacci sequence of order 1, and the ordinary Fibonacci sequence is called a Fibonacci sequence of order 2, is there a formula for finding a Fibonacci sequence of order n that will satisfy a Fibonacci Nim game where $q_m = n \cdot r_{m-1}$? Dr. Gaskell and the author have worked on this problem independently and have found two different methods of determining an order n Fibonacci sequence. All of the sequences investigated so far take the form of $f_i = f_{i-1} + f_{i-p}$, but as of yet no relationship has been found between the order of the sequence and p.

## REFERENCES

1. Richard L. Frey (ed.), The New Complete Hoyle (Philadelphia, David McKay Company, 1947) pp. 705 — 706.
2. Charles L. Bouton, "Nim, A Game with a Complete Mathematical Theory," Annals of Mathematics, 1901 — 1902, pp. 35 — 39.

## Comments

1. While the proof makes use only of the fact that every Fibonacci number is at least as large as its immediate predecessor and of the recurrence property, $f_n = f_{n-1} + f_{n-2}$, it should be noted that Lucas numbers cannot be substituted for Fibonacci numbers, because the number 2 cannot be represented in a Lucas number system using only 1's and 0's. One might define $L_1$ as 2 and $L_2$ as 1 in order to make a Lucas number system, but this would invalidate the required property that every member of the sequence is at least as large as its predecessor.

2. (Editorial Comment) The uniqueness follows from Zeckendorf's Theorem. If the Fibonacci numbers $u_1, u_2, \cdots$ are defined by $u_1 = 1$, $u_2 = 2$, $u_n = u_{n-1} + u_{n-2}$, $n \geq 3$.

Theorem. For each natural number N there is one and only one system of natural numbers $i_1, i_2, \cdots i_d$ such that

$$N = u_{i_1} + u_{i_2} + \cdots + u_{i_d} \text{ and } i_{\nu+1} \geq i_\nu + 2 \text{ for } 1 \leq \nu < d.$$

3. This is an example of how the Fibonacci number system can be used to prove theorems about Fibonacci numbers. The example shown is a generalized form of the theorems concerning the sum of odd or even Fibonacci numbers. Another simple example is to find the sum of the Fibonacci numbers through $f_n$. One simply represents all the Fibonacci numbers through an arbitrary n, 5 for example, in the Fibonacci number system: $11111_f$. $11111_f = 10101_f + 1010_f$. Since $10101_f = 100000_f - 1_{ten}$ and $1010_f = 10000_f - 1_{ten}$, $11111_f = 110000_f - 2$, or $1000000_f - 2$. In other words, the sum of the Fibonacci numbers through $f_n = f_{n+2} - 2$.

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