FIBONACCI REPRESENTATIONS OF HIGHER ORDER

* L. CARLITZ and RICHARD SCOVILLE
Duke University, Durham, North Carolina
and
V. E. HOGGATT, JR.
San Jose State College, San Jose, California

1. INTRODUCTION

Let

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1) .$$

It is well known that every positive integer N can be uniquely represented in the form

(1.1) $$N = F_{k_1} + F_{k_2} + F_{k_3} + \cdots ,$$

where

(1.2) $$k_1 \geq 2, \quad k_{i+1} - k_i \geq 2 \qquad (i = 1, 2, 3, \cdots) ;$$

Equation (1.1) is called the canonical representation of N. Let A_k denote the set of positive integers N with $k_1 = k$ in (1.1). It was proved in [2] that

(1.3) $$A_{2t} = ab^{t-1}a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots) ,$$

(1.4) $$A_{2t+1} = b^t a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots) ,$$

where $\mathbb{N}$ denotes the set of positive integers and the functions a(n), b(n) are defined by means of

(1.5) $$a(n) = [\alpha n], \quad b(n) = [\alpha^2 n], \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}) ,$$

*Supported in part by NSF Grant GP-17071.

and $[x]$ denotes the greatest integer $\leq x$. In the paper cited, considerable use is made of the function $e(N)$ defined by

$$e(N) = F_{k_1-1} + F_{k_2-1} + F_{k_3-1} + \cdots ; \tag{1.6}$$

This function was introduced in an earlier paper [1].

It is natural to try to extend the results of [2] to Fibonacci numbers of higher order. For a number of reasons we limit ourselves in the present paper to the numbers defined by

$$G_0 = 0, \quad G_1 = G_2 = 1, \quad G_{n+1} = G_n + G_{n-1} + G_{n-2} \quad (n \geq 2). \tag{1.7}$$

To begin with, we have the unique canonical representation

$$N = \epsilon_2 G_2 + \epsilon_3 G_3 + \epsilon_4 G_4 + \cdots , \tag{1.8}$$

where each ϵ_i is either 0 or 1 and now

$$\epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 0 \quad (i = 2, 3, 4, \cdots). \tag{1.9}$$

Corresponding to the function $e(N)$ defined by (1.6) we introduce the function

$$f(N) = \epsilon_2 G_1 + \epsilon_3 G_2 + \epsilon_4 G_3 + \cdots . \tag{1.10}$$

Moreover if

$$N = \epsilon_2' G_2 + \epsilon_3' G_3 + \epsilon_4' G_4 + \cdots , \tag{1.11}$$

where each ϵ_i' is either 0 or 1, is any representation of N, then

$$f(N) = \epsilon_2' G_1 + \epsilon_3' G_2 + \epsilon_4' G_4 + \cdots .$$

Let C_k denote the set of positive integers $\{N\}$ for which ϵ_k is the first nonzero ϵ_i in (1.8). We obtain results analogous to (1.3) and (1.4), namely

(1.12) $$C_{3k+2} = ac^k a(\mathbf{N}) \cup ac^k b(\mathbf{N}) \qquad (k \geq 0),$$

(1.13) $$C_{3k+3} = bc^k a(\mathbf{N}) \cup bc^k b(\mathbf{N}) \qquad (k \geq 0),$$

(1.14) $$C_{3k+4} = c^{k+1} a(\mathbf{N}) \cup c^{k+1} b(\mathbf{N}) \qquad (k \geq 0).$$

The functions a, b, c are defined in Section 3 below; we have been unable to find explicit formulas analogous to (1.5). We show, however, that the functions can be characterized in the following way. They are strictly monotone functions whose ranges constitute a disjoint partition of the positive integers; moreover

(1.15) $$b(n) = a^2(n) + 1, \qquad c(n) = a(n) + b(n) + n.$$

In addition to the canonical representation (1.8), we find it convenient to introduce a second canonical representation

(1.16) $$N = G_{3k+1} + \epsilon_{3k+2} G_{3k+2} + \cdots,$$

where $k \geq 0$ and as before

$$\epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 0 \qquad (i \geq 3k + 1).$$

Moreover, making use of the representation (1.16),

(1.17) $$\begin{cases} a(N) = G_{3k+2} + \epsilon_{3k+2} G_{3k+3} + \cdots \\ b(N) = G_{3k+3} + \epsilon_{3k+2} G_{3k+4} + \cdots \\ c(N) = G_{3k+4} + \epsilon_{3k+2} G_{3k+5} + \cdots \end{cases}$$

It is because of these formulas for a(N), b(N), c(N) that (1.16) is particularly useful.

2. PRELIMINARIES

Let Q_n be the set of non-negative N's which can be written canonically in the form

$$N = \epsilon_2 G_2 + \epsilon_3 G_3 + \cdots + \epsilon_n G_n . \tag{2.1}$$

Then we have

$$\begin{aligned} Q_2 &= \{0, 1\}, \qquad Q_3 = \{0, 1, 2, 3\}, \\ Q_4 &= \{0, 1, 2, 3, 4, 5, 6, \cdots\} \end{aligned} \tag{2.2}$$

We can see easily by induction that Q_n is a disjoint union:

$$Q_n + (Q_{n-3} + G_{n-1} + G_n) \cup (Q_{n-2} + G_n) \cup Q_{n-1} \tag{2.3}$$

and that

$$Q_n = \{0, 1, 2, \cdots, G_{n+1} - 1\} . \tag{2.4}$$

These remarks imply the following theorem.

Theorem 1. Any positive integer N can be uniquely represented in the canonical form (2.1).

Theorem 2. If N is given (not necessarily canonically) by

$$N = \epsilon_2' G_2 + \epsilon_3' G_3 + \cdots ,$$

then

$$f(N) = \epsilon_2' G_1 + \epsilon_3' G_2 + \cdots .$$

Proof. Given any representation $\epsilon' = (\epsilon_2', \epsilon_3', \cdots)$ of N we obtain another representation $s(\epsilon')$ of N by choosing, in ϵ, the block of the form (1, 1, 1, 0) that is farthest right and replacing it by the block (0, 0, 0, 1). If there is no such block, ϵ' is canonical and we set $s(\epsilon') = \epsilon'$. It is clear that sufficiently many applications of s will yield the canonical representation of N, but it is also clear that

$$\epsilon_2' G_1 + \epsilon_3' G_2 + \cdots = s(\epsilon')_2 G_1 + s(\epsilon')_3 G_2 + \cdots , \tag{2.5}$$

establishing the theorem.

Theorem 3. We have $f(N + 1) \geq f(N)$, with equality if and only if $N \in C_2$.

Proof. If $N \notin C_2$ then

$$N = \epsilon_3 G_3 + \epsilon_4 G_4 + \cdots$$

and

$$N + 1 = G_2 + \epsilon_3 G_3 + \cdots .$$

Hence

$$f(N + 1) = G_1 + \epsilon_3 G_2 + \cdots = f(N) + 1 .$$

If $N \in C_2$ then either

(a) $$N = G_2 + G_3 + \epsilon_5 G_5 + \cdots$$

or

(b) $$N = G_2 + \epsilon_4 G_4 + \cdots .$$

In case (a)

$$N + 1 = G_4 + \epsilon_5 G_5 + \cdots$$

and

$$f(N + 1) = G_3 + \epsilon_5 G_4 + \cdots = f(N) .$$

In case (b)

$$N + 1 = G_3 + \epsilon_4 G_4 + \cdots$$

and

$$f(N+1) = G_2 + \epsilon_4 G_3 + \cdots = f(N) .$$

This completes the proof.

Theorem 4. We have $N - 1 \notin C_2$ if and only if $N \in C_k$, where $k \equiv 2 \pmod 3$.

Proof. If $N \in C_2$, there is nothing to prove, so suppose $N \in C_k$, $k > 2$; let N have the canonical representation

$$N = G_k + \epsilon_{k+1} G_{k+1} + \cdots .$$

Then we have

$$G_k = \begin{cases} G_0 + G_1 + G_2 + (G_4 + G_5) + \cdots + G_{k-2} + G_{k-1} & k \equiv 0 \pmod 3 \\ G_1 + G_2 + G_3 + (G_5 + G_6) + \cdots + (G_{k-2} + G_{k-1}) & k \equiv 1 \pmod 3 \\ G_2 + G_3 + G_4 + (G_6 + G_7) + \cdots + (G_{k-2} + G_{k-1}) & k \equiv 2 \pmod 3). \end{cases} \tag{2.6}$$

Thus we see that only in the case $k \equiv 2 \pmod 3$ we have $G_k - 1 \notin C_2$.

Theorem 5. The following identities hold for $k > 2$.

$$f(G_k - 1) = \begin{cases} G_{k-1} & k \equiv 0 \pmod 3) \\ G_{k-1} & k \equiv 1 \pmod 3) \\ G_{k-1} - 1 & k \equiv 2 \pmod 3) \end{cases} \tag{2.7}$$

Proof. Making use of (2.6), we readily get (2.7).

3. THE FUNCTIONS a, b, AND c

In this section we define three strictly monotone functions on the positive integers, which we display as an array:

(3.1) R :

1	2	3	4	5	
a(1)	a(2)	a(3)	a(4)	a(5)	···
b(1)	b(2)	b(3)	b(4)	b(5)	···
c(1)	c(2)	c(3)	c(4)	c(5)	···

We begin by setting $a(1) = 1$, $b(1) = 2$, $c(1) = 4$, $a(2) = 3$, and fill the rest of the array by induction. Suppose that columns 1 to n have been filled, and also that $a(n + 1)$ is known. Then we fill row a to column $a(n + 1)$ in increasing order with the first integers that have not appeared so far in the array. Then we let $b(n + 1)$ be the next integer that has not appeared, and we set

$$c(n + 1) = n + 1 + a(n + 1) + b(n + 1) .$$

Thus we get

(3.2) R :

n	1	2	3	4	5	6	7	8	9	10
a	1	3	5	7	8	10	12	14	16	18
b	2	6	9	13	15	19	22	26	30	
c	4	11	17	24	28	35	41	48	55	

It is clear from the definition of R that the ranges $a(N)$, $b(N)$, and $c(N)$ are disjoint and exhaust the positive integers. We will now establish several relations between a, b and c.

Theorem 6. For every positive integer N, the following identities hold:

$$c(N) = a(N) + b(N) + N , \tag{3.3}$$

$$b(N) = a^2(N) + 1 , \tag{3.4}$$

$$ab(N) = ba(N) + 1 , \tag{3.5}$$

$$c(N) = ab(N) + 1 = ba(N) + 2 . \tag{3.6}$$

Proof. 1. ((3.3)) This is the definition of $c(N)$.

2. ((3.4)) Let N be the first integer for which (3.4) fails. Then we must have, for some $K < N$,

$$c(K) = a^2(N) + 1 \qquad \text{and} \qquad b(N) - a^2(N) + 2 .$$

Hence the array has the form

K	N	a(N)
.	.	$a^2(N)$
.	$a^2(N) + 2$	
$a^2(N) + 1$		

Now $K + N + a(N)$ numbers have been entered. Since they must be the numbers $1, 2, \cdots, a^2(N) + 2$, we get

$$K + N + a(N) = a^2(N) + 2 . \tag{3.7}$$

But

$$K + a(K) + b(K) = c(K) = a^2(N) + 1 .$$

Therefore

$$a(K) + b(K) + 1 = N + a(N) . \tag{3.8}$$

Now if we had $a(K) < N$, we would have $a^2(K) < a(N)$, but from (3.8) we would have $b(K) + 1 > a(N)$. However $b(K) = a^2(K) + 1$ since (3.4) holds for $K < N$. This is a contradiction, since $a(N) < b(K)$. In a similar way we contradict the supposition $a(K) > N$. Hence $a(K) = N$ and we have

$$K + N + a(N) = K + a(K) + a^2(K) = K + a(K) + b(K) - 1 = c(K) - 1 = a^2(N),$$

contradicting (3.7).

3. ((3.5) and (3.6)). Consider the array:

(3.9)

N	a(N)	$a^2(N)$	b(N)
.	$a^2(N)$	ba(N) - 1	ab(N)
.	ba(N)		
c(N)			

Assume $ba(N) > c(N)$. Then no number $\leq c(N)$ can be missing from the enclosed portion (since it's too late to enter it in any row). Hence in the enclosed portion we have at least the numbers $1, 2, \cdots, c(N)$, $ba(N) - 1$ and $ba(N)$. However these are only $N + a(N) + b(N) - 1 = c(N) - 1$ entries, a contradiction. Hence $ba(N) < c(N)$ and one number $M < c(N)$ is missing from the enclosed portion. Then we must have $M = ab(N)$. Now M is exceeded only by $c(N)$, so we must have $ab(N) = c(N) - 1$, $ba(N) = c(N) - 2$ proving (3.5) and (3.6).

We conclude this section with a characterization of the array R.

Theorem 7. Let a_1, b_1 and c_1 be strictly monotone functions whose ranges form a disjoint partition of the positive integers. Suppose further that they satisfy (3.3) and (3.4). Then $a_1 = a$, $b_1 = b$ and $c_1 = c$.

Proof. Clearly

$$b(N) = a^2(N) + 1 > a^2(N) \geq a(N) .$$

Hence we must have $a(1) = 1$ and $b(1) = a^2(1) + 1 = 2$. Then $c(1) = 4$, and further, since $b(N) > a(N)$, $a(2) = 3$.

Now by induction on the columns of the array formed by the functions a_1, b_1 and c_1, we see that it is the array R.

4. RELATIONS INVOLVING f

Since every number appears in the range of f and f is monotone, the following definition makes sense. For every N, we let A(N) be defined as follows:

$$(4.1) \qquad f(A(N)) = N; \qquad f(A(N) - 1) = N - 1 .$$

We define B(N) by

$$B(N) = A(A(N)) + 1 \tag{4.2}$$

and C(N) by

$$C(N) = N + A(N) + B(N) . \tag{4.3}$$

Theorem 8. $A(A(\mathbf{N})) \subseteq C_2$.

Proof. Suppose for some N, A(A(N)) is not in C_2. Put (canonical representation)

$$A(A(N)) = G_k + \epsilon_{k+1} G_{k+1} + \cdots \qquad (k > 2) .$$

Then applying f we get

$$\begin{aligned} A(N) &= G_{k-1} + \epsilon_{k+1} G_k + \cdots \\ N &= G_{k-2} + \epsilon_{k+1} G_{k-1} + \cdots \end{aligned} \tag{4.4}$$

By the definition of A and Theorem 3, $A(A(N)) - 1 \notin C_2$, so by Theorem 4, $k \equiv 2 \pmod 3$. But neither is $A(N) - 1$ in C_2. Hence $k - 1 \equiv 2 \pmod 3$. This is a contradiction and proves the theorem.

Theorem 9. $C_2 = A(A(\mathbf{N})) \cup A(B(\mathbf{N}))$.

Proof. Suppose $A(B(N)) \notin C_2$. Put (canonical representation)

$$A(B(N)) = G_k + \epsilon_{k+1} G_{k+1} + \cdots \qquad (k > 2) .$$

As in the previous theorem, we must have $k \equiv 2 \pmod 3$ so that $k \geq 5$ and

$$A(A(N)) + 1 = B(N) = G_{k-1} + \epsilon_{k+1} G_k + \cdots .$$

Hence

$$A(A(N)) = G_{k-1} - 1 + \epsilon_{k+1} G_k + \cdots$$

and, from Theorem 5,

$$A(N) = G_{k-2} + \epsilon_{k+1} G_{k-1} + \cdots .$$

Now again since $A(N) - 1 \notin C_2$, we get $k - 2 \equiv 2 \pmod 3$, a contradiction.

Theorem 10. Let K be arbitrary and suppose $K - 1$ is given canonically by

$$K - 1 = \epsilon_2 G_2 + \epsilon_3 G_3 + \cdots . \tag{4.5}$$

Then

$$A(K) = G_1 + \epsilon_2 G_3 + \epsilon_3 G_4 + \cdots \tag{4.6}$$

$$A(A(K)) = G_2 + \epsilon_2 G_4 + \epsilon_3 G_5 + \cdots \tag{4.7}$$

$$C(K) = G_4 + \epsilon_2 G_5 + \epsilon_3 G_6 + \cdots . \tag{4.8}$$

Proof. From the previous theorem, the number

$$P = G_2 + \epsilon_2 G_4 + \epsilon_3 G_5 + \cdots$$

is either of the form $A(A(L))$ or $A(B(L))$. Hence

$$f(P) = G_1 + \epsilon_2 G_3 + \epsilon_3 G_4$$

is either of the form $A(L)$ or $B(L)$. But $f(P) - 1 \notin C_2$, so $f(P)$ cannot have the form $B(L)$. Hence P is, in fact, $A(A(K))$ and all of the relations follow, the third using (4.2) and (4.3).

Theorem 11. $A = a$, $B = b$, $C = c$ and for any integer N,

$$\begin{cases} f(a(N)) = N \\ f(b(N)) = a(N) \\ f(c(N)) = b(N) . \end{cases} \tag{4.9}$$

Proof. We prove the first part of the theorem by verifying the conditions of Theorem 7. The second part will be established incidentally in the course of the proof. Only the requirement that $A(N)$, $B(N)$ and $C(N)$ be disjoint and exhaustive is not clear.

Now $A(A(N)) \in C_2$ by Theorem 8, so by Theorem 3,

$$f(B(N) - 1) = f(A(A(N))) = f(B(N)) .$$

Hence $B(N) \notin A(N)$ (and $f(B(N)) = A(N)$). Now

$$C(N) = N + A(N) + A(A(N)) + 1 .$$

Let (canonical representation)

$$A(A(N)) = G_2 + \epsilon_3 G_3 + \epsilon_4 G_4 + \cdots .$$

Then

$$A(N) - 1 = \epsilon_3 G_2 + \epsilon_4 G_4 + \cdots ,$$

and, since $A(N) - 1 \notin C_2$, it follows that $\epsilon_3 = 0$. Applying f we get

$$N - 1 = \epsilon_3 G_1 + \epsilon_4 G_2 + \cdots ,$$

so that

$$\begin{aligned} C(N) &= 3 + F_2 + \epsilon_3 G_4 + \epsilon_4 G_5 + \cdots \\ &= G_4 + \epsilon_4 G_5 + \epsilon_5 G_6 + \cdots . \end{aligned}$$

This is not necessarily the canonical representation of $C(N)$ but

$$f(C(N)) = A(A(N)) + 1 \qquad (= B(N))$$

and

$$f(C(N) - 1) = A(A(N)) + 1 .$$

Hence $C(N) \notin A(\mathbf{N})$. Now suppose $C(N) = A(A(M)) + 1$ for some M. Then

$$B(N) = A(A(N)) + 1 = f(C(n)) = A(M) ,$$

a contradiction. Hence we have shown that $A(\mathbf{N})$, $B(\mathbf{N})$ and $C(\mathbf{N})$ are disjoint.

Now suppose $N \notin A(\mathbf{N}) \cup B(\mathbf{N})$. Let (canonical representation)

$$N = G_k + \epsilon_{k+1} G_{k+1} + \cdots . \tag{4.10}$$

By Theorem 9 this is equivalent to assuming $A(N) \notin C_2$, that is,

$$A(N) = G_{k+1} + \epsilon_{k+1} G_{k+2} + \cdots , \tag{4.11}$$

and since, always, $A(N) - 1 \notin C_2$, we have $k + 1 \equiv 2 \pmod 3$, that is, $k \equiv 1 \pmod 3$.

First let us consider the case $k = 4$. Then, if we put

$$K - 1 = \epsilon_5 G_2 + \epsilon_6 G_3 + \cdots , \tag{4.12}$$

we get, by Theorem 10,

$$c(K) = N .$$

Now suppose $k > 4$; $k = 3t + 1$, $t > 1$. Then let $s = t - 1$ and set

$$\begin{aligned} K &= G_{3s+1} + \epsilon_{3t+2} G_{3s+2} + \cdots \\ &= G_{-2} + (G_{-1} + G_0) + (G_2 + G_3) + \cdots + (G_{3s-1} + G_{3s}) + \epsilon_{3t-2} G_{3s+2} \cdot \end{aligned}$$

Now, applying Theorem 10 to $K - (G_{-2} + (G_{-1} + G_0))$, we get

$$\begin{aligned} &C(K - (G_{-2} + G_{-1} + G_0) + 1) \\ &\quad = G_4 + (G_5 + G_6) + \cdots + (G_{3t-1} + G_{3t}) + \epsilon_{3t+2} G_{3t+2} + \cdots \\ &\quad = N. \end{aligned}$$

This proves the Theorem.

5. THE SECOND CANONICAL REPRESENTATION

Theorem 12. Every positive integer N can be written in a unique way in the form

$$N = G_{3s+1} + \epsilon_{3s+2} G_{3s+2} + \cdots \tag{5.1}$$

where $s \geq 0$ and, as before, $\epsilon_i \epsilon_{i+1} \epsilon_{i+2} = 0$. Moreover,

$$a(N) = G_{3s+2} + \epsilon_{3s+2} G_{3s+3} + \cdots , \tag{5.2}$$

$$b(N) = G_{3s+3} + \epsilon_{3s+2} G_{3s+4} + \cdots , \tag{5.3}$$

and

$$c(N) = G_{3s+4} + \epsilon_{3s+2} G_{3s+5} + \cdots . \tag{5.4}$$

Proof. We saw in the proof of the previous theorem that an integer M is of the form c(K) if and only if it is given canonically by

$$M = G_k + \epsilon_{k+1} G_{k+1} + \cdots , \qquad k \equiv 1 \pmod 3.$$

Hence for some $s \geq 0$, c(N) is given canonically by

$$c(N) = G_{3s+4} + \epsilon_{3s+2} G_{3s+5} + \cdots .$$

Apply f repeatedly to get the existence of the representation and formulas (5.2) and (5.3). Now if we assume that N can be written in two different ways in the form (5.1), we should obtain two different canonical representations of c(N). Hence the theorem is proved.

We may call (5.1) the second canonical representation.

In view of the representation (5.1), it is natural to let C_{3s+1} denote the set of integers representable in the form (5.1), for a fixed s. Then clearly

(5.5) $$\bar{C}_{3s+1} = C_{3s+1} \qquad (s \geq 1)$$

while

(5.6) $$\bar{C}_1 = \bigcup_{k=0}^{\infty} (C_{3k+2} \cup C_{3k+3})$$

Making use of the last theorem, we obtain several formulas relating a, b and c. The details are similar in all cases so we will prove only two of the formulas.

Theorem 13. The following formulas hold.

(5.7) $$\begin{cases} a^2 = b - 1 \\ ab = c - 1 \\ ac = a + b + c \\ ba = c - 2 \\ b^2 = a + b + c - 1 \\ bc = a + 2b + 2c \\ ca = a + b + c - 3 \\ cb = a + 2b + 2c - 2 \\ c^2 = 2a + 3b + 4c \end{cases}$$

Proof. To prove, for instance, that

$$bc = a + 2b + 2c ,$$

we suppose that

(5.8) $$c(N) = G_{3s+1} + \epsilon_{3s+2} G_{3s+2} + \cdots \qquad (s \geq 1) .$$

Then by Theorem 12

$$bc(N) = G_{3s+3} + \epsilon_{3s+2} G_{3s+4} + \cdots \qquad (s \geq 1) .$$

But, by applying f to c(N) we see that

$$b(N) = G_{3s} + \epsilon_{3s+2} G_{3s+1} + \cdots \qquad (s \geq 1)$$

and

$$a(N) = G_{3s-1} + \epsilon_{3s+2} G_{3s} + \cdots \qquad (s \geq 1) .$$

Now the result follows if we observe that

$$G_{n-1} + 2G_n + 2G_{n-1} = G_{n+3} .$$

Similarly, to prove that

$$b^2 = a + b + c - 1 ,$$

suppose that

$$c(N) = G_{3s+1} + \epsilon_{3s+2} G_{3s+2} + \cdots \qquad (s \geq 1) .$$

Then

$$b(N) = G_{3s} + \epsilon_{3s+2} G_{3s+1} + \cdots \qquad (s \geq 1)$$

and

$$a(N) = G_{2s-1} + \epsilon_{3s+2} G_{3s} + \cdots \qquad (s \geq 1) .$$

Now we write b(N) in the second canonical form:

$$b(N) = (G_1 + G_2) + \cdots + (G_{3s-2} + G_{3s-1}) + \epsilon_{3s+2} G_{3s+1} + \cdots \quad .$$

Then

$$b^2(n) = (G_3 + G_4) + \cdots + (G_{3s} + G_{3s+1}) + \epsilon_{3s+2} G_{3s+3} + \cdots .$$

Hence

$$b^2(N) + 1 = b^2(N) + G_2 = a(N) + b(N) + c(N) .$$

A word function (or simply word) u is a monomial in a, b, c:

$$u = a^{i_1} b^{j_1} c^{k_1} \cdots a^{i_r} b^{j_r} c^{k_r} , \tag{5.9}$$

where the exponents are arbitrary nonnegative integers. Since $au = bv$, for example, is impossible, and $au = av$ implies $u = v$, it follows that the representation (5.9) is unique. In other words factorization into prime elements a, b, c is unique. We define the weight of a word by means of

$$p(a) = 1, \qquad p(b) = 2, \qquad p(c) = 3$$

together with

$$p(uv) = p(u) + p(v) ,$$

where u, v are arbitrary words. Let N_p denote the number of words of weight p. If u is any such word then either

$$u = au_1, \qquad u = bu_2 \qquad \text{or} \qquad u = cu_3 ,$$

where

$$p(u_1) = p - 1, \qquad p(u_2) = p - 2, \qquad p(u_3) = p - 3 .$$

Hence

$$N_p = N_{p-1} + N_{p-2} + N_{p-3} \qquad (p \geq 3) .$$

Moreover

$$N_0 = N_1 = 1, \qquad N_2 = 2 .$$

It follows that

$$N_p = G_{p+1} . \tag{5.10}$$

Theorem 14. The words u, v satisfy

$$uv = vu \tag{5.11}$$

if and only if there is a word w such that

$$u = w^r, \qquad v = w^s ,$$

where r and s are nonnegative integers.

Proof. The proof is by induction on $p(u) + p(v)$. We may assume that both u, v have positive weight. Also we may assume that $p(u) \geq p(v)$. It then follows from (5.11) and unique factorization that $u = vz$, where z is a word. Thus (5.11) reduces to

$$zv = vz . \tag{5.12}$$

Since $p(zv) < p(uv)$, the inductive hypothesis gives

$$z = w^r, \qquad v = w^s ,$$

so that $u = w^{r+s}$.

For the next theorem we require, in addition to the weight of u, the degree of u, d(u), defined by

$$d(u) = i_1 + j_1 + k_1 + \cdots + i_r + j_r + k_r , \tag{5.13}$$

where u is given by (5.9). We also define the integers H_n by means of

(5.14) $H_0 = 0, \quad H_1 = 1, \quad H_2 = 2, \quad H_{n+1} = H_n + H_{n-1} + H_{n-2} \quad (n \geq 2).$

It is easy to show that

(5.15) $$H_n = G_n + G_{n-1} .$$

Theorem 15. Let u be a word of weight p. Then

(5.16) $$u(n) = G_{p-3}a(n) + H_{p-3}b(n) + G_{p-2}c(n) = \lambda_u ,$$

where λ_u is independent of n but depends on u.

To have the theorem hold for all $p \geq 1$ we extend the definition of G_n, H_n for negative values of n. In particular, we have the following table of values

n	-3	-2	-1	0	1	2	3
G	-1	1	0	0	1	1	2
H	-1	0	1	0	1	2	3

It is now easily verified that the theorem holds for the words a, b and c. We assume that (5.16) holds for words of degree k. Let u be an arbitrary word of degree k + 1. There are three cases according as u = va, vb or vc. Assume v has weight p.

(i) For u = va, we have, by the inductive hypothesis and Theorem 13,

$$\begin{aligned} u = va &= G_{p-3}a^2 + H_{p-3}ba + g_{p-2}ca - \lambda_v \\ &= G_{p-3}(b - 1) + H_{p-3}(c - 2) + G_{p-2}(a + b + c - 3) - \lambda_v \\ &= G_{p-2}a + (G_{p-2} + G_{p-3})b + (H_{p-3} + G_{p-2})c - (G_{p-3} + 2H_{p-3} + 3G_{p-2} + \lambda_v) \\ &= G_{p-2}a + H_{p-2}b + G_{p-1}c - (H_p + \lambda_v) . \end{aligned}$$

(ii) For u = vb, we have

$$
\begin{aligned}
u = vb &= G_{p-3}ab + H_{p-3}b^2 + G_{p-2}cb - \lambda_v \\
&= G_{p-3}(c - 1) + H_{p-3}(a + b + c - 1) + G_{p-2}(a + 2b + 2c - 2) - \lambda_v \\
&= (G_{p-2} + H_{p-3})a + (2G_{p-2} + H_{p-3})b + (2G_{p-2} + G_{p-3} + H_{p-3})c \\
&\qquad - (2G_{p-2} + G_{p-3} + H_{p-3} + \lambda_v) \\
&= G_{p-1}a + H_{p-1}b + G_pc - (G_p + \lambda_v) .
\end{aligned}
$$

(ii) For $u = vc$, we have

$$
\begin{aligned}
u = vc &= G_{p-3}ac + H_{p-3}bc + G_{p-2}c^2 - \lambda_v \\
&= G_{p-3}(a + b + c) + H_{p-3}(a + 2b + 2c) + G_{p-2}(2a + 3b + 4c) = \lambda_v \\
&= (2G_{p-2} + G_{p-3} + H_{p-3})a + (3G_{p-2} + G_{p-3} + 2H_{p-3})b \\
&\qquad + (4G_{p-2} + G_{p-3} + 2H_{p-3})c = \lambda_v \\
&= G_pa + H_pb + G_{p+1}c - \lambda_v .
\end{aligned}
$$

This completes the proof. Incidentally, we have proved the following relations:

$$
\text{(5.17)} \qquad \begin{cases} \lambda_{va} = H_p + \lambda_v \\ \lambda_{vb} = G_p + \lambda_v \\ \lambda_{vc} = \lambda_v \end{cases} ,
$$

where v is of weight p.

As an immediate corollary of the last theorem, we state:

Theorem 16. Let u and v be arbitrary words. Then there is an integer C such that

$$
\text{(5.18)} \qquad uv - vu = C .
$$

6. AN ESTIMATE OF $a(n)$

Let α be the real root of $x^3 - x^2 - x - 1 = 0$ and let β and γ be the complex roots, $\beta = re^{i\theta}$, $\gamma = re^{-i\theta}$. Then we have

(6.1) $$G_{n+1} - \alpha G_n = \frac{\gamma^{n+1} - \beta^{n+1}}{\gamma - \beta}$$

which we can verify by taking n equal to -1, 0 and 1 and noting that both sides of (6.1) satisfy the recurrence

$$u_{n+3} = u_{n+2} + u_{n+1} + u_n .$$

If N is given in the second canonical representation by

(6.2) $$N = G_{3k+1} + \epsilon_{3k+2} G_{3k+2} + \cdots \qquad (k \geq 0),$$

we have

(6.3) $$\begin{cases} a(N) = G_{3k+2} + \epsilon_{3k+2} G_{3k+3} + \cdots \\ b(N) = G_{3k+3} + \epsilon_{3k+2} G_{3k+4} + \cdots \\ c(N) = G_{3k+4} + \epsilon_{3k+2} G_{3k+5} + \cdots . \end{cases}$$

Now $\alpha = 1.8, \cdots$, so that $|\beta| = |\gamma| = \sqrt{\beta\gamma} = \sqrt{1/\alpha} < 1$. Then, using (6.1), (6.2), and (6.3) we get the following.

Theorem 17. The three sequences

$$a(N) - [\alpha N], \qquad b(N) - [\alpha^2 N], \qquad c(N) - [\alpha^3 N]$$

are all bounded.

Next we prove

Theorem 18. The difference $a(N) - [\alpha N]$ is positive infinitely often, negative infinitely often and 0 infinitely often.

Proof. If θ were a rational multiple of 2π we should have, for some m,

$$\gamma^{m+1} = \beta^{m+1} = r^{m+1}$$

and, by (6.1), $G_{m+1} = \alpha G_m$. But α is irrational so this is impossible. Hence for infinitely many k we must have

$$(6.4) \qquad G_{3k+2} - \alpha G_{3k+1} = r^{3k+1} \frac{\sin\{(3k+2)\theta\}}{\sin\theta} > 0 ,$$

that is,

$$a(G_{3k+1}) - [\alpha G_{3k+1}] > 0 ,$$

for infinitely many k.

To get the second part of the theorem we must find an infinite number of integers N for which

$$a(N) - \alpha N < -1 .$$

Let N have the form

$$N = G_1 + G_3 + G_k ,$$

where k is very large. Then

$$\begin{aligned} a(N) - \alpha N &= G_2 - \alpha G_1 + G_4 - \alpha G_3 + G_{k+1} - \beta G_k \\ &= 1 - \alpha + 3 - 2\alpha + G_{k+1} - \alpha G_k \approx -1.4 . \end{aligned}$$

This proves the theorem.

Finally to prove that the difference vanishes infinitely often, it suffices to show that (compare (6.4))

$$-1 \le r^{3k+1} \frac{\sin(3k+2)}{\sin\theta} < 0$$

for infinitely many values of n. This is clear since $0 < r < 1$ and θ is an irrational multiple of 2π.

7. GENERATING FUNCTIONS

Put

$$\phi_k(x) = \sum_{n \in C_k} x^n \qquad (k = 2, 3, 4, \cdots) \tag{7.1}$$

and

$$\overline{\phi}_{3k+1}(x) = \sum_{n \in \overline{C}_{3k+1}} x^n \qquad (k = 0, 1, 2, \cdots). \tag{7.2}$$

In view of (5.5) and (5.6), we have

$$\overline{\phi}_{3k+1}(x) = \phi_{3k+1}(x) \qquad (k = 1, 2, 3, \cdots) \tag{7.3}$$

and

$$\overline{\phi}_1(x) = \sum_{k=0}^{\infty} \phi_{3k+2}(x) + \sum_{k=0}^{\infty} \phi_{3k+3}(x) . \tag{7.4}$$

It is evident that

$$\frac{x}{1 - x} = \sum_{k=2}^{\infty} \phi_k(x) . \tag{7.5}$$

Also it follows from the definition of C_k that

$$\begin{aligned} \phi_k(x) &= x^{G_k}\left\{1 + \sum_{j=k+2}^{\infty} \phi_j(x)\right\} \\ &\quad + x^{G_k + G_{k+1}}\left\{1 + \sum_{j=k+3}^{\infty} \phi_j(x)\right\} . \end{aligned} \tag{7.6}$$

From (1.1) we get the recurrence

$$
\begin{aligned}
(7.7)\qquad &(1 + x^{G_{k+1}})(\phi_{k+1}(x) + x^{G_{k+1}+G_{k+2}}\phi_{k+3}(x)) \\
&= x^{G_{k+1}-G_k}(1 + x^{G_{k+2}})(\phi_k(x) - x^{G_k}\phi_{k+2}(x)) .
\end{aligned}
$$

It is also convenient to define

$$
(7.8)\qquad A(x) = \sum_{n=1}^{\infty} x^{a(n)}, \qquad B(x) = \sum_{n=1}^{\infty} x^{b(n)}, \qquad C(x) = \sum_{n=1}^{\infty} x^{c(n)} ,
$$

so that

$$
(7.9)\qquad A(x) + B(x) + C(x) = \frac{x}{1 - x} .
$$

Moreover

$$
(7.10)\qquad A(x) = \sum_{k=0}^{\infty} \phi_{3k+2}(x) ,
$$

$$
(7.11)\qquad B(x) = \sum_{k=0}^{\infty} \phi_{3k+3}(x) ,
$$

$$
(7.12)\qquad C(x) = \sum_{k=0}^{\infty} \phi_{3k+4}(x) = \frac{x}{1 - x} - \bar{\phi}_1(x) .
$$

Now by (7.1) and (1.12)

$$
\phi_2(x) = \sum_{n=1}^{\infty} x^{a^2(n)} + \sum_{n=1}^{\infty} x^{ab(n)} .
$$

Since

$$a^2(N) = b(n) - 1, \qquad ab(n) = c(n) - 1 ,$$

It follows that

$$x\phi_2(x) = B(x) + C(x) . \tag{7.13}$$

In the next place, by (1.13)

$$\phi_3(x) = \sum_{n=1}^{\infty} x^{ba(n)} + \sum_{n=1}^{\infty} x^{b^2(n)}$$

$$= x^{-2}C(x) + x^{-1}\sum_{n=1}^{\infty} x^{a(n)+b(n)+c(n)} .$$

Since

$$A(x) = \sum_{n=1}^{\infty} x^{a^2(n)} + \sum_{n=1}^{\infty} x^{ab(n)} + \sum_{n=1}^{\infty} x^{ac(n)}$$

$$= x^{-1}A(x) + x^{-1}B(x) + \sum_{n=1}^{\infty} x^{a(n)+b(n)+c(n)} ,$$

it follows that

$$x^2\phi_3(x) = xA(x) - B(x) . \tag{7.14}$$

By (1.1)

$$\begin{aligned}\phi_2(x) &= x\left\{1 + \sum_{j=4}^{\infty}\phi_j(x)\right\} + x^3\left\{1 + \sum_{j=5}^{\infty}\phi_j(x)\right\} \\ &= x + x^3 + x\phi_4(x) + (x + x^3)\sum_{j=5}^{\infty}\phi_j(x) \\ &= x + x^3 + x\phi_4(x) + (x + x^3)\left\{\frac{x}{1 - x} - \phi_2(x) - \phi_3(x) - \phi_4(x)\right\}.\end{aligned}$$

Combining this with (7.13) and (7.14), we get

$$x^4\phi_4(x) = x^2n(x) - C(x) . \tag{7.15}$$

In a similar manner we get

$$x^8\phi_5(x) = -xA(x) + B(x) + (1 + x^4)C(x) , \tag{7.16}$$

$$x^{15}\phi_6(x) = (x + x^8)A(x) - (1 + x^2 + x^7)B(x) - x^7C(x) . \tag{7.17}$$

Generally it can be shown that

$$x^{S_{k-1}}\phi_k(x) = p_1(x)A(x) + p_2(x)B(x) + p_3(x)C(x) , \tag{7.18}$$

where

$$S_k = G_1 + G_2 + \cdots + G_k$$

and $p_1(x)$, $p_2(x)$, $p_3(x)$ are polynomials with integral coefficients.

In the next place, exactly as in [1, Sec. 7], we can show that $A(x)$, $B(x)$ and $C(x)$ cannot be continued analytically across the unit circle. For the proof it suffices to use

$$\sum_{a(k)\le n} 1 \sim \frac{n}{\alpha}, \quad \sum_{b(k)\le n} 1 \sim \frac{n}{\alpha^2}, \quad \sum_{c(k)\le n} 1 \sim \frac{n}{\alpha^3} . \tag{7.19}$$

which follow from Theorem 17. Indeed, we can show in this way that none of the functions can be continued across the unit circle. Moreover if we put

$$\phi_k(x) = \phi_k^{a}(x) + \phi_k^{(b)}(x) ,$$

where (compare (1.12), (1.13), (1.14))

$$\phi^a_{3k+2}(x) = \sum_{n=1}^{\infty} x^{ac^k a(n)}, \quad \phi^a_{3k+3}(x) = \sum_{n=1}^{\infty} x^{bc^k a(n)},$$

$$\phi_{3k+4}(x) = \sum_{n=1}^{\infty} x^{c^{k+1} a(n)}, \quad \phi^b_{3k+2}(x) = \sum_{n=1}^{\infty} x^{ac^k b(n)}, \qquad \text{and}$$

$$\phi^b_{3k+3}(x) = \sum_{n=1}^{\infty} x^{bc^k b(n)}, \quad \phi^b_{3k+4}(x) = \sum_{n=1}^{\infty} x^{c^{k+1} b(n)} ,$$

then neither $\phi_k^a(x)$ nor $\phi_k^b(x)$ can be continued across the unit circle.

We can also show that $A(x)$, $B(x)$, $C(x)$ do not satisfy any relation of the form

$$f_1(x)A(x) + f_2(x)B(x) + f_3(x)C(x) = 0 , \tag{7.20}$$

where $f_1(x)$, $f_2(x)$, $f_3(x)$ are polynomials. In the first place we may assume without loss of generality that the coefficients of $f_1(x)$ are rational (for proof compare [3, p. 141, No. 151]) and that

$$(f_1(x), \quad f_2(x), \quad f_3(x)) = 1 . \tag{7.21}$$

Since (7.19) implies

$$\lim_{x=1} (1 - x)A(x) = \frac{1}{\alpha} \qquad \lim_{x=1} (1 - x)B(x) = \frac{1}{\alpha^2}, \qquad \lim_{x=1} (1 - x)C(x) = \frac{1}{\alpha^3},$$

[Continued on page 94.]