# FIBONACCI REPRESENTATIONS OF HIGHER ORDER - II

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## 1. INTRODUCTION

Let $N \geq 2$ be a fixed integer. We wish to discuss various properties of sequences $\{v_n\}$ $(n = 0, \pm 1, \pm 2, \cdots)$ of complex numbers satisfying the recurrence

$$(1.1) \qquad v_{n+N} = v_{n+N-1} + \cdots + v_{n+1} + v_n \qquad (n = 0, \pm 1, \pm 2, \cdots) .$$

We let $\mathbb{V}$ be the set of sequences satisfying (1.1) and we let $\mathbb{D}$ be the set of all sequences $\delta_n$ $(n = 0, \pm 1, \pm 2, \cdots)$ which are non-zero on only a finite number of coordinates. For $\delta \in \mathbb{D}$ and $v \in \mathbb{V}$ we define

$$\delta(v) = \sum \delta_n v_n \quad .$$

We will call $\delta \in \mathbb{D}$ <u>canonical</u> if

$$(1.2) \qquad \delta_i \neq 0 \Rightarrow \delta_i = 1 \qquad (i = 0, \pm 1, \cdots)$$

and

$$(1.3) \qquad \delta_i \delta_{i+1} \cdots \delta_{i+N-1} = 0 \qquad (i = 0, \pm 1, \cdots) .$$

We will say $\epsilon$ and $\epsilon' \in \mathbb{D}$ are <u>equivalent</u> $(\epsilon \equiv \epsilon')$ if $\epsilon(v) = \epsilon'(v)$ for all $v \in R$.

We shall also have occasion to use the translation operator $T$ on sequences from $\mathbb{D}$ or $\mathbb{V}$ defined by

$$(1.4) \qquad (Tv)_n = v_{n+1} \qquad (v \in \mathbb{D} \text{ or } \mathbb{V}) .$$

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The main theorem of the present paper is the following.

Theorem A. Let $\epsilon \in \mathbf{D}$ have integral coordinates. Then either $\epsilon$ or $-\epsilon$ is equivalent to a canonical element of $\mathbf{D}$.

We use this theorem first to generalize a result of Klarner's [4] for Fibonacci numbers to $N^{th}$ order Fibonacci numbers $P = \{P_n\}$ defined by

(i) $$P \in \mathbf{W}$$

(ii) $$P_{-(N-2)} = \cdots = P_0 = 0, \quad P_1 = 1.$$

The generalization is as follows:

Theorem B. Let $K_1, K_2, \cdots, K_N$ be positive integers. Then there is a unique canonical $\delta \in \mathbf{D}$ such that

$$K_i = \delta(T^iP) \qquad (i = 1, 2, \cdots, N)\,. \tag{1.5}$$

If $\gamma$ is a root of

$$x^N - x^{N-1} - \cdots - x - 1 = 0 \tag{1.6}$$

we let $\underline{\gamma}$ be the sequence in $\mathbf{W}$ defined by

$$(\underline{\gamma})_n = \gamma^n\,. \tag{1.7}$$

We let $\alpha$ be the largest positive root of (1.6). Note that $\alpha > 1$.

As a corollary to the main theorem we get

Theorem C. A positive real number x is of the form $\delta(\underline{\alpha})$ for some canonical $\delta \in \mathbf{D}$ if and only if, for some positive k and some integers $Q_1$, $Q_2, \cdots, Q_N$ we have

$$\alpha^k x = Q_1 + Q_2\alpha + \cdots + Q_N\alpha^{N-1}\,. \tag{1.8}$$

In Section 4, we assume that $N = 3$ and verify some conjectures of Hoggatt concerning certain functions introduced and discussed in [1], [2] and

[3]. The authors believe that the results obtained in Section 4 for the case $N = 3$ are strongly indicative of those that might hold for larger values of N.

## 2. PROPERTIES OF CANONICAL ELEMENTS

Theorem 1. Suppose $\delta$ and $\epsilon \in \mathbb{D}$ are canonical. Then either $\delta - \epsilon$ or $\epsilon - \delta$ is equivalent to $\gamma \in \mathbb{D}$.

Proof. The non-zero coordinates of $\eta = \delta - \epsilon$ are 1's and -1's. Suppose the first non-zero coordinate of $\eta$ (starting from the left) is -1, and let $\eta_k = 1$ be the first 1. Now change $\eta_k$ to 0 and add 1 to each of $\eta_{k-1}, \eta_{k-2}, \cdots, \eta_{k-N}$. The resulting sequence is equivalent to $\eta$, and since $\delta$ and $\epsilon$ are canonical, it can be seen that not all of $\eta_{k-1} + 1, \cdots, \eta_{k-N} + 1$ are 0. Performing this "change" repeatedly, we finally come to a sequence $\eta'$ equivalent to $\eta$ all of whose non-zero coordinates are either 1 or -1. This of course implies that either $\eta$ or $-\eta$ is equivalent to a canonical element of $\mathbb{D}$.

Theorem 2. Let $\epsilon \in \mathbb{D}$ have integral coordinates. Then either $\epsilon$ or $-\epsilon$ is equivalent to a canonical element of $\mathbb{D}$. If the coordinates of $\epsilon$ are non-negative then $\epsilon$ is equivalent to a canonical element of $\mathbb{D}$.

Proof. We set $\epsilon = \epsilon^+ - \epsilon^-$. The previous theorem shows that the first statement of the present theorem follows from the second; so we assume $\epsilon = \epsilon^+$.

Now a simple induction shows that it is enough to prove the following statement: If $\epsilon$ is canonical, then $\epsilon + \chi_i$ is equivalent to a canonical element, where $\chi_i$ is defined by

$$\chi_i(V) = v_i \qquad v \in \mathbb{V} . \tag{2.1}$$

Note that $\epsilon + \chi_i = \epsilon - \chi_{i-1} - \cdots - \chi_{i-N+1} + \chi_{i+1} \equiv \gamma_1 + \chi_{i+1}$ where, by Theorem 1 either $\gamma_1$ or $-\gamma_1$ is canonical. If $-\gamma_1$ is canonical, then again by Theorem 1, $\gamma_1 + \chi_{i+1}$ is equivalent to a canonical element. Hence we may suppose $\gamma_1$ is canonical. Then we get

$$\epsilon + \chi_i \equiv \gamma_1 + \chi_{i+1} \equiv \gamma_2 + \chi_{i+2} \equiv$$

with $\gamma_1, \gamma_2, \cdots$ canonical. But this is impossible for, if so, we would have

$$(2.2)\qquad [\epsilon + \chi_i](\underline{\alpha}) \geq \chi_{i+n}(\underline{\alpha}) = \alpha^{i+n} \qquad (n = 1, 2, \cdots) .$$

This completes the proof.

Let $P \in \mathbf{V}$ be the sequence defined by the initial conditions

$$(2.3)\qquad P_{-(N-2)} = \cdots = P_0 = 0; \qquad P_1 = 1 .$$

Theorem 3. Let K be a positive integer. Then there is a unique canonical $\delta \in \mathbf{D}$ such that, for all n,

$$(2.4)\qquad P_n K = \sum_i \delta_i P_{i+n} .$$

Proof. Let $\epsilon \in \mathbf{D}$ be the sequence

$$(2.5)\qquad \epsilon_n = \begin{cases} K & n = 0 \\ 0 & \text{otherwise} \end{cases} .$$

Then by Theorem 2 there is a unique canonical $\delta \in \mathbf{D}$ satisfying

$$(2.6)\qquad \epsilon(v) = \delta(v), \qquad v \in \mathbf{V} .$$

Letting v be translates of P we get (2.4) immediately since $\epsilon(v) = v_0 K$ for any $v \in \mathbf{V}$.

The uniqueness of $\delta$ will follow if we can show that any $\gamma \in \mathbf{V}$ is determined by its value on translates of P. We state this as a separate theorem.

Theorem 4. $\mathbf{V}$ is N-dimensional as a complex vector space. It is spanned by P, TP, $\cdots$, $T^{N-1}P$. Moreover, the $N \times N$ matrix

$$\Delta_i = \{(T^j P)_n\} \qquad \begin{matrix} (j = 0, 1, \cdots, N-1) \\ (n = 0, i+1, \cdots, i+N-1) \end{matrix}$$

has determinant

$$|\Delta_i| = \left((-1)^{N+1}\right)^{i+1} .$$

Proof. The fact that V is N-dimensional is well-known, so the calculation of the determinant will complete the proof: we have

$$\Delta_i = \begin{pmatrix} P_i & P_{i+1} & \cdots & P_{i+n-1} \\ P_{i+1} & P_{i+2} & \cdots & P_{i+n} \\ \vdots & & & \\ P_{i+n-1} & & \cdots & P_{i+2n-2} \end{pmatrix}$$

Adding the last N - 1 columns to the first, using the recurrence and interchanging columns we get

$$|\Delta_i| = (-1)^{N+1}|\Delta_{i+1}| . \tag{2.7}$$

But

$$\Delta_{-(N-2)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 1 & \cdots & & & \\ 1 & 1 & \cdots & & & \end{pmatrix}$$

so that

$$|\Delta_{-(N-2)}| = (-1)^{N+1} .$$

Hence

$$|\Delta_i| = \left((-1)^{N+1}\right)^{i+1} .$$

Theorem 5. Let $v \in \mathbf{V}$. Then

$$(2.8)\quad v = v_0TP + (v_1 - v_0)P + (v_2 - v_1 - v_0)T^{-1} + \cdots + (v_{N-1} + \cdots - v_1 - v_0)T^{-(N-2)}P \ .$$

Proof. Let $0 \le j \le N - 1$. The $j^{th}$ coordinate of the right side is

$$\begin{aligned} v_0P_{j+1} + (v_1 + v_0)P_j + \cdots + (v_{N-1} - \cdots - v_1 - v_0)P_{j-(N-2)} & \\ = v_0(P_{j+1} - P_j - \cdots - P_{j-(N-2)}) & \\ + v_1(P_j - P_{j-1} - \cdots - P_{j-(N-2)}) & \\ + v_k(P_{j+1-k} - \cdots - P_{j-(N-2)}) & \\ \vdots \qquad & \\ + v_{N-2}(P_{j-(N-2)+1} - P_{j-(N-2)}) & \\ + v_{N-1}(P_{j-(N-2)}) & \ . \end{aligned}$$

The coefficient of $v_k$ is non-zero only when $j + 1 - k = 1$, i.e., only when $k = j$. In this case it is 1.

We can generalize a theorem proved by Klarner for the Fibonacci numbers as follows.

Theorem 6. Let $K_1, K_2, K_3, \cdots, K_N$ be positive integers. Then there is a unique canonical $\delta$ such that

$$(2.10)\qquad K_i = \delta(T^iP) \qquad (i = 1, 2, \cdots, N) \ .$$

Proof. It will be enough to find a canonical $\delta$ satisfying

$$(2.11)\qquad K_i = \delta\left(T^{i-(N-1)}P\right) \qquad (i = 1, 2, \cdots, N)$$

because then a translate of $\delta$ will satisfy (2.10). Let $\gamma$ be one of the N roots of $x^N - x^{N-1} - \cdots - x - 1 = 0$, and let

$$(2.12)\qquad v = \underline{\gamma} \ .$$

Then by the previous theorem, if $\delta$ exists and satisfies (2.11) it must also satisfy

$$\delta(\underline{\gamma}) = K_N + (\gamma - 1)K_{N-1} + \cdots + (\gamma^{N-1} - \gamma^{N-2} - \cdots - \gamma - 1)K_1$$

$$(2.13) \qquad = \frac{1 + \gamma + \cdots + \gamma^{N-1}}{\gamma^N} K_N + \frac{1 + \gamma + \cdots + \gamma^{N-2}}{\gamma^{N-1}} + \cdots + \frac{1}{\gamma} K_1$$

$$= K_N\gamma^{-N} + (K_N + K_{N-1})\gamma^{-(N-1)} + \cdots + (K_N + \cdots + K_1)\gamma^{-1} .$$

Hence we should define $\delta$ to be the unique canonical form in $\mathbb{D}$ equivalent to $\beta \in \mathbb{D}$ where $\beta$ is given by

$$(2.14) \qquad \beta_i = \begin{Bmatrix} K_N + \cdots + K_i & (-N \le i \le -1) \\ 0 & (\text{otherwise}) \end{Bmatrix}$$

Now

$$(2.15) \qquad \beta\left(T^{i-(N-1)}P\right) = \sum_{j=1}^{N} (K_N + \cdots + K_j)P_{-j+i-(N-1)}$$

$$= \sum_{t=1}^{N} K_t\left(\sum_{j=1}^{t} P_{-j+i-(N-1)}\right) = K_i .$$

## 3. FURTHER APPLICATIONS OF THE MAIN THEOREM

We recall that $\alpha$ is the largest positive root of

$$x^N - x^{N-1} - \cdots - x - 1 = 0$$

and

$$\underline{\alpha} = (\ldots, \alpha^{-1}, 1, \alpha, \cdots) .$$

<u>Theorem 7</u>. Let K be any positive integer. Then there exists a unique canonical $\delta \in \mathbb{D}$ such that

$$K = \delta(\underline{\alpha}) .$$

Moreover,

$$K = \delta(P) .$$

Proof. Choose $\delta$ as in Theorem 3. Then

$$\delta(\underline{\alpha}) = \epsilon(\underline{\alpha}) = K .$$

Theorem 8. A positive real number $x$ is of the form $\delta(\underline{\alpha})$ for some canonical $\delta \in \mathbb{D}$ if and only if, for some positive $k$ and some integers $Q_1$, $Q_2, \cdots, Q_N$ we have

$$\alpha^k x = Q_1 + Q_2 \alpha + \cdots + Q_N \alpha^{N-1} . \tag{3.1}$$

Proof. Suppose first that $x$ is of the form $\delta(\underline{\alpha})$:

$$x = \sum_{j=-k} \epsilon_j \alpha^j . \tag{3.2}$$

Then

$$\alpha^k x = \sum_{j=0} \epsilon_j \alpha^{j+k}$$

and powers of $\alpha$ higher than $\alpha^{N-1}$ can be successively reduced to lower powers eventually giving (3.1).

Now suppose (3.1) holds. Let $\epsilon \in \mathbb{D}$ be defined by

$$\epsilon_n = \begin{cases} Q_{n+k+1} & -k \leq n \leq N-k-1 \\ 0 & \text{otherwise} \end{cases} . \tag{3.3}$$

Then either $\epsilon$ or $-\epsilon$ is equivalent to a canonical element $\delta \in \mathbb{D}$. But

$$\epsilon(\alpha) = x > 0 .$$

Hence we must have $\epsilon \equiv \delta$.

4.

For the notation used in the remainder of the paper we refer the reader to [3].

Let $\nu_k(M)$ denote the number of numbers $n \in C_k$ such that $n \le M$.

Theorem 9. If $M \notin C_2$ then

$$\nu_2(M) = M - f(M) . \tag{4.1}$$

More generally, if

$$M \notin C_2 \cup C_3 \cup \cdots \cup C_r$$

then

$$\nu_r(M) = f^{r-2}(M) - f^{r-1}(M) \qquad (r = 2, 3, 4, \cdots) . \tag{4.2}$$

Proof. Let

$$K_r = \{K | K \notin C_2 \cup C_3 \cup \cdots \cup C_r\} , \qquad r \ge 2$$

and let $K_1 = \mathbb{N}$. Then clearly $f^{r-1}$ is 1 - 1, onto and monotone from $K_r$ to $\mathbb{N}$. In particular,

$$\text{card}\,\{K | K \in K_r,\ K \le M\} = f^{r-1}(M) \qquad (r = 1, 2, \cdots) . \tag{4.3}$$

Hence

$$\nu_r(M) = \text{card}\,\{K|K \in C_r;\ K < M\} = \text{card}\,\{K|K \in K_{r-1},\ K \le M\}$$
$$- \text{card}\,\{K|K \in F_r,\ K \le M\} = f^{r-2}(M) - f^{r-1}(M).$$

The following theorem is an immediate corollary.

Theorem 10. We have

$$\nu_2(G_n) = G_n - G_{n-1} = G_{n-2} + G_{n-3} \qquad (n \geq 3)\,. \tag{4.4}$$

More generally

$$\nu_r(G_n) = G_{n-r+2} - G_{n-r+1} = G_{n-r} + G_{n-r-1} \qquad (n \geq r+1)\,. \tag{4.5}$$

Theorem 11. Let $k$ and $r$ be fixed integers, $k \geq 1$, $r \geq 2$. Then

$$\nu_r(kG_n) = k(G_{n-r} + G_{n-r-1}) \tag{4.6}$$

for $n$ sufficiently large.

Proof. Using Theorem 3, we let $\delta \in \mathbf{D}$ be canonical such that

$$kG_n = \sum \delta_i G_{i+n}, \qquad (n = 0, 1, 2, \cdots)\,. \tag{4.7}$$

Hence for $n$ sufficiently large we will have

$$kG_n \notin C_2 \cup \cdots \cup C_r\,,$$

so

$$\begin{aligned} \nu_r(kG_n) &= f^{r-2}(kG_n) - f^{r-1}(kG_n) \\ &= \sum \delta_i G_{i+n-(r-2)} - \sum \delta_i G_{i+n-(r-1)} \\ &= kG_{n-(r-2)} - kG_{n-(r-1)} \\ &= k(G_{n-r} + G_{n-r-1})\,. \end{aligned} \tag{4.8}$$

The last three theorems were conjectured by Hoggatt.

REFERENCES

1. L. Carlitz, V. E. Hoggatt, Jr., and Richard Scoville, "Fibonacci Representations," Fibonacci Quarterly, Vol. 10 (1972), pp. 1-28.

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